Colourful components in k-caterpillars and planar graphs

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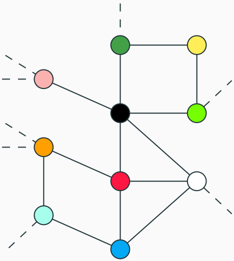
Colourful components, colourful graphs and bad paths

We consider undirected graphs with colored vertices.



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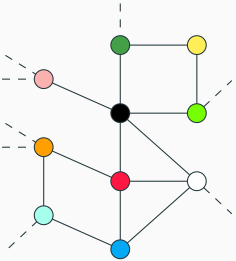
Some definitions

A *colourful component* is a *connected component* whose vertices have different colours.



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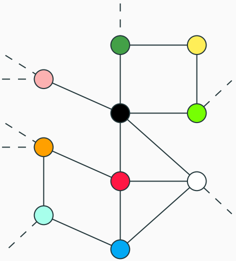
A graph is *colourful* if all its connected components are colourful.



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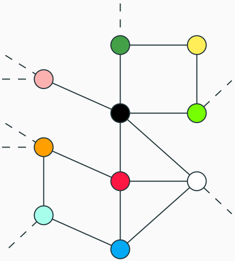
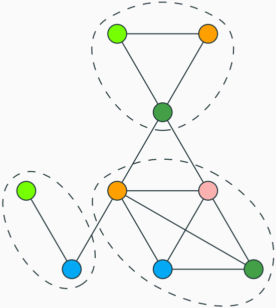
A *bad path* is a path with endpoints of the same colour.



Our problems

- COLOURFUL COMPONENTS:
  Are there at most p edges whose removal makes the graph colourful?
- COLOURFUL PARTITION:
  Is there a partition of the graph with at most p parts such that each part is a colourful component?



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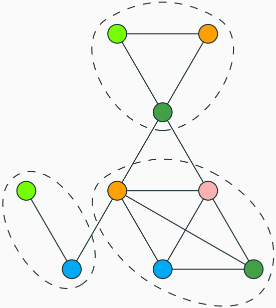

Motivation

Partition a set of genes in *orthologous genes*, which are *sets of genes from different species* that have evolved through speciation events only.

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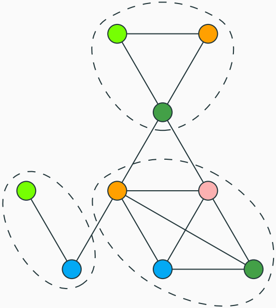

Observation

COLOURFUL COMPONENTS and COLOURFUL PARTITION are linear-time equivalent on forests.

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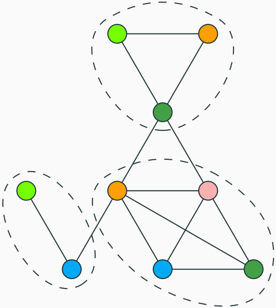


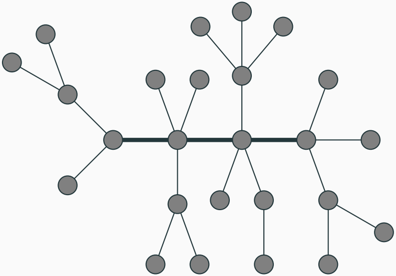
Observation

Colourful Components and Colourful Partition are linear-time equivalent on forests.

We can only consider connected graphs.

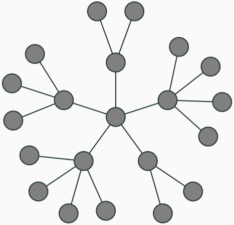
k-caterpillars

A *k-caterpillar* is a tree in which all the vertices are within distance k of a central path, called the *backbone*.



Example of a 2-caterpillar.

Known results on trees with bounded diameter



Example of a tree with diameter 4.

Theorem (Bruckner *et al.*, '12)

COLOURFUL COMPONENTS is NP-*complete on trees with diameter* 4.

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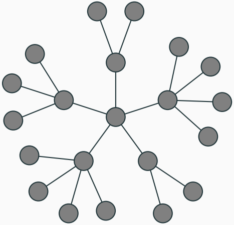

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Corollary

COLOURFUL COMPONENTS is NP-*complete on* 2-*caterpillars.*

Known results on trees with bounded degree

Theorem (Dondi and Sikora, '18)

COLOURFUL COMPONENTS is solvable in time $\mathcal{O}(n^2)$ on paths with n vertices (trees with max. degree ≤ 2).

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Colourful Components is NP*-complete on trees with maximum degree* 6.

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Colourful Components is NP-*complete on trees with maximum degree* 6.

The complexity of Colourful Components on trees with maximum degree ≤ 5 was left open in the same paper.

Our goal

We would like complexity dichotomies for COLOURFUL COMPONENTS on trees with respect to:

- the maximum degree d of the input tree,

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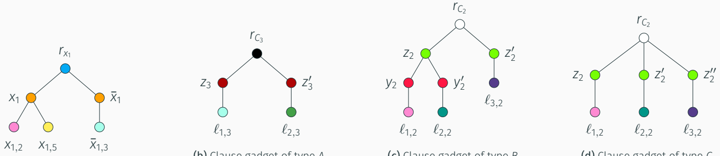
- the maximum degree d of the input tree,
- the smallest integer k such that the input tree is a k-caterpillar,
- both d and k.

Bad news!

Theorem (Chlebíková and D.)
Colourful Components is NP*-complete on:*

- *4-caterpillars with maximum degree* ≤ 3*,*
- *3-caterpillars with maximum degree* ≤ 4*, and*
- *2-caterpillars with maximum degree* ≤ 5*.*



(a) Variable gadget of x_1
(b) Clause gadget of type A of the clause C_3
(c) Clause gadget of type B of the clause C_2
(d) Clause gadget of type C of the clause C_2

Reduction from 3,3-SAT (≤ 3 var. per clause, ≤ 3 occurences var.).

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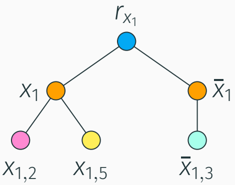
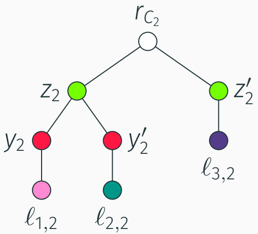


Variable gadget of x_1



Clause gadget of C_2 of type B of the clause C_2

But pseudocaterpillars are fine

Definition

A *cyclic k-caterpillar* is a connected graph G with a unique cycle B, called the backbone, such that for any $e \in E(B)$, the graph $G - e$ is a k-caterpillar.

A *k-pseudocaterpillar* is either a k-caterpillar or a cyclic k-caterpillar.

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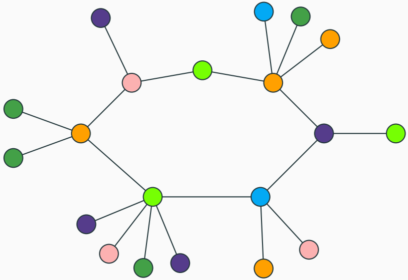
Theorem (Chlebíková and D.)

Colourful Components is can be solved in linear time on 1-pseudocaterpillars.

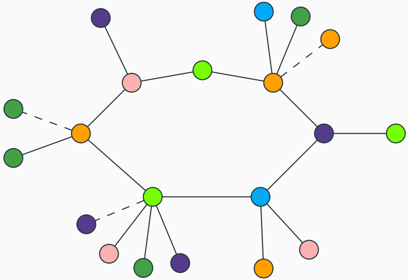
Algorithm’s idea

We can see the graph as a collection of stars that are connected via their internal vertices (in a path or cycle manner).



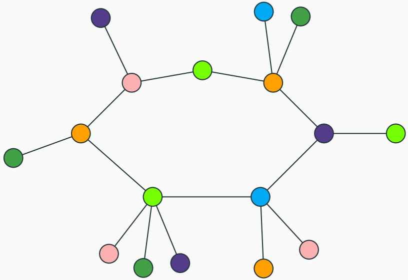
Algorithm’s idea

We preprocess the graph, store the removed edges and focus on the component containing the backbone.



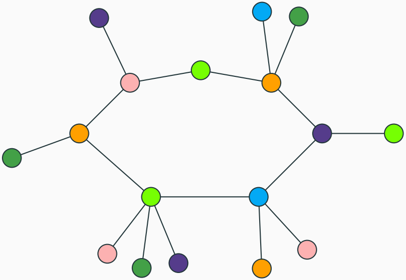
Algorithm's idea

We obtain that every star is colourful.



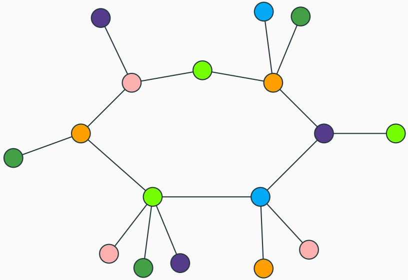
Algorithm's idea

We can show that there exists an optimal solution (a set of edges) for which every edge belongs to the backbone.



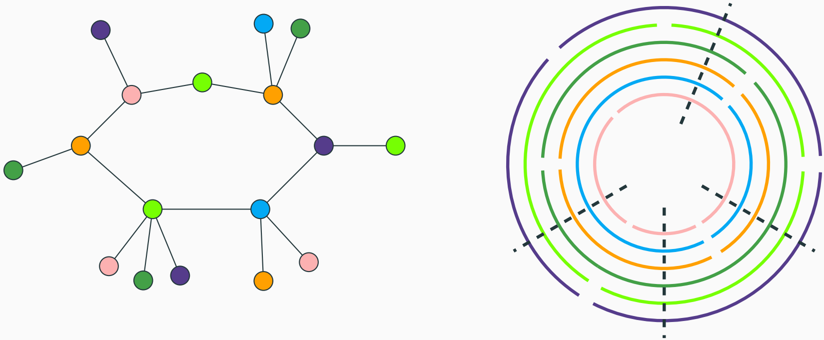
Algorithm's idea

It is in fact enough to consider a specific kind of bad paths: the *critical bad paths*.



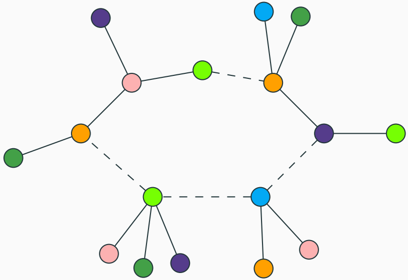
Algorithm's idea

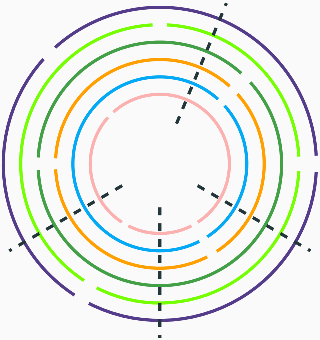
We represent the *critical bad paths* as arcs on the circle and compute in linear time (Hsu and Tsai, '91) a minimum clique cover on the circular-arc graph.



Algorithm's idea

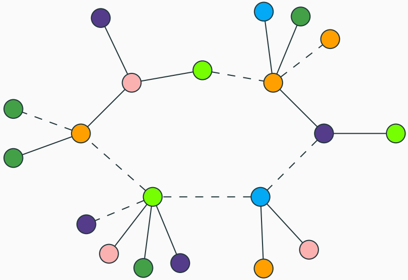
For each clique in the clique cover, we remove one corresponding edge in the backbone. These edges intersect every critical bad path.





Algorithm’s idea

Eventually, we return the union of the edges removed in the preprocessing and the newly obtained. This is an optimal solution.



Recap

- NP-complete on 4-caterpillars with max. degree ≤ 3.
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- linear-time solvable on 1-pseudocaterpillars.

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So we obtain two complexity dichotomies on trees:

- Colourful Components is linear-time solvable on trees with maximum degree d if $d \leq 2$, and NP-complete otherwise.
- Colourful Components is linear-time solvable on k-caterpillars if $k \leq 1$, and NP-complete otherwise.

A complexity dichotomy in terms of d and k?

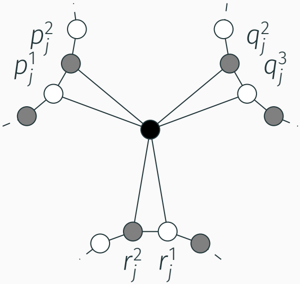
The complexity of Colourful Components is still open on:

- 2-caterpillars with max. degree ≤ 4,
- 2-caterpillars with max. degree ≤ 3,
- 3-caterpillars with max. degree ≤ 3.

Known results on planar graph with bounded degree

Theorem (Bruckner *et al.*, '12)

Colourful Components is NP-*complete on* 3-*coloured planar graphs with maximum degree* 6.



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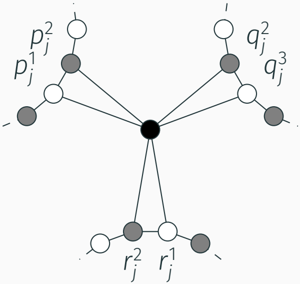
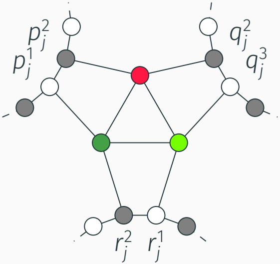


Question: Can the maximum degree be decreased while preserving NP-completeness (and bounded number of colours)?

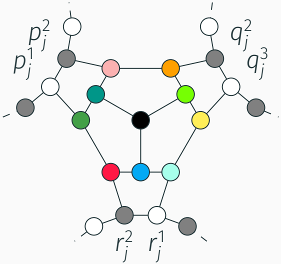
Planar graphs with even smaller degrees (but more colours)

Theorem (Chlebíková and D.)

Colourful Components is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3.



Gadget with vertices of degree 4.



Gadget with vertices of degree 3.

This time, we reduce from Planar 3-SAT.

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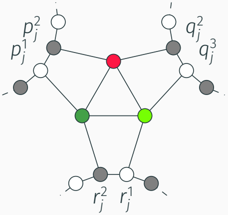

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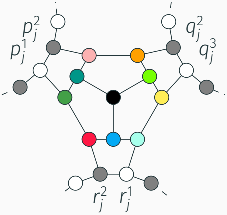

Gadget with vertices of degree 3.

There might still be room for improvement...
Eventually, we would like a complexity dichotomy based on the maximum degree and number of colours (in planar graphs).

Thank you!