Generalised periodic solutions to a forced Kepler problem in the plane

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Periodically forced Kepler problem

Find T-periodic solutions to:

$$(FKP\epsilon) \qquad \ddot{x} = -\frac{x}{|x|^3} + \epsilon \nabla_x U(t,x) \quad x \in \mathbb{R}^2 \setminus \{O\}$$

where:

- $U : \mathbb{R}^{1+2} \to \mathbb{R}$ smooth enough;
- $U(t+T,x) = U(t,x)$ for all $(t,x) \in \mathbb{R}^{1+2}$ and some $T > 0$.

As critical points in $\mathcal{H}^1_T := \{x \in \mathcal{H}^1(0,T) : x(0) = x(T)\}$ of

$$\mathcal{A}_T(x) = \int_0^T \left(\frac{|\dot{x}(t)|^2}{2} + \frac{1}{|x(t)|} + \epsilon U(t,x(t)) \right) dt$$

There are $x \in \mathcal{H}^1_T$ such that $O \in x([0,T])$ and $\mathcal{A}_T(x) < +\infty$.

Related papers: collisionless solutions

For $(FKP\epsilon)$:

- Ambrosetti & Coti Zelati 1989: U even and $T/2$ periodic;
- Cabral & Vidal, 2000: U symmetric under rotation and reflection;
- Fonda & Toader & Torres, 2012;
- Fonda & Gallo, 2017: radial perturbation, 2018: symmetry under a rotation;
- Boscaggin & Ortega, 2016: averaging technique;
- Amster & Haddad & Ortega & Ureña 2011: large perturbations;

For (FKP):

- Serra & Terracini 1994: $U(t,x) = p(t)$ ruled out.

Related papers: generalised solutions

- Solutions attaining O on a zero-measure set: Ambrosetti & Coti Zelati 1993, Bahri & Rabinowitz, Tanaka 1993;
- regularised equations in dimension 1: Ortega 2011, Zhao 2016, Rebelo & Simões 2018;
- regularised equations in higher dimension: Boscaggin & Ortega & Zhao 2019;
- regularised functionals in general setting: Barutello & Ortega & Verzini 2021.

Generalised solutions

A generalised T-periodic solution to (FKP) is a T-periodic function $x \in C(\mathbb{R})$ that satisfies the following:

1. the collision set $E_x := x^{-1}(O) = \{t \in [0, T] : x(t) = O\}$ is discrete;
2. $x \in C^2(I)$ and satisfies equation (FKP) in I, for each interval $I \subset \mathbb{R} \setminus E_x$;
3. the limits:

$$\lim_{t \to t_0} \frac{x(t)}{|x(t)|} \quad \text{and} \quad \lim_{t \to t_0} \left(\frac{|\dot{x}(t)|^2}{2} - \frac{1}{|x(t)|} \right)$$

exist and are finite at every $t_0 \in E_x$.

Result

Theorem

If $U(t,x)$ is $C^1(\mathbb{R}^{1+2})$, T-periodic w.r.t. t and satisfies:

$$|U(t,x)| \leq C(1+|x|^{\alpha}) \quad \forall (t,x) \in \mathbb{R}^{1+2}$$

for some $C > 0$ and $\alpha \in]0,2[$, then (FKP) has at least one T-periodic generalised solution.

Candidates are chosen among the local minimisers of the action functional

$$\mathcal{A}_T(x) = \int_0^T \left(\frac{|\dot{x}(t)|^2}{2} + \frac{1}{|x(t)|} + U(t,x(t)) \right) dt$$

which lacks coercivity on $\mathcal{H}^1_T := \{x \in \mathcal{H}^1(0,T) : x(0) = x(T)\}$.

Minimisation

We consider $\mathcal{X} := \mathcal{X}_c \cup \mathcal{X}_r$ where:

- $\mathcal{X}_c := \left\{ x \in \mathcal{H}^1_T : O \in x([0,T]) \right\}$;
- $\mathcal{X}_r := \left\{ x \in \mathcal{H}^1_T : O \notin x([0,T]) \text{ and } x \text{ is not null-homotopic in } \mathbb{R}^2 \setminus \{O\} \right\}$.
- $\mathcal{X}$ is sequentially weakly closed in $\mathcal{H}^1_T$.
- A Poincaré-type inequality holds in $\mathcal{X}$:

$$\int_0^T |x|^2 \leq K \int_0^T |\dot{x}|^2 \quad \forall x \in \mathcal{X} \quad \Longrightarrow \quad \mathcal{A}_T(x) \geq \int_0^T \left(\frac{|\dot{x}|^2}{4} + \frac{|x|^2}{8K} \right) - K' \quad \forall x \in \mathcal{X}.$$

Proposition

There exists $x \in \mathcal{X}$ *such that* $\mathcal{A}_T(x) = \inf_{y \in \mathcal{X}} \mathcal{A}_T(y)$.

From now on, we assume that $x \in \mathcal{X}_c$.

Exploring collisions

- The collision set $E_x = x^{-1}(O) \subset [0, T]$ has measure 0 since $\mathcal{A}_T(x) \in \mathbb{R}$;
- $\mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}}(E_x + kT)$ is the (at most) countable union of pairwise disjoint open intervals $]a_n, b_n[$ where x is $C^2(]a_n, b_n[)$ and satisfies (FKP) $(n \in \mathbb{N})$.
- if we let

$$h_x(t) = \frac{|\dot{x}(t)|^2}{2} - \frac{1}{|x(t)|} \qquad t \in [0, T] \setminus E_x,$$

we have that

$$\int_0^T |h_x(t)|dt \leq \mathcal{A}_T(x) - \int_0^T U(t, x(t))dt \Longrightarrow h_x \in L^1(0, T).$$

Exploring collisions: the energy

Proposition

$h_x \in W^{1,1}_{loc}$ and, therefore, the energy can be extended to a continous function.

- choose any $\phi \in C_c^\infty(0,T)$ and define $\psi_\lambda(t) = t + \lambda\phi(t)$ and let $x_\lambda = x \circ \psi_\lambda$;
- if λ is small enough, ϕ_λ is a diffeomorphism, $x_\lambda([0,T]) = x([0,T])$ and, in particular, $x_\lambda \in \mathcal{X}_c$;
- if $a(\lambda) := \mathcal{A}_T(x_\lambda)$, then $a(\lambda) \geq a(0) = \mathcal{A}_T(x)$ for each λ in a neighborhood of 0 and, thus, $a'(0) = 0$;
- more precisely:

$$\int_0^T \left[h_x(t)\dot\phi(t) + \langle \nabla_x U(t,x(t)), \dot x(t) \rangle \phi(t) \right] dt = 0 \quad \forall \phi \in C_c^\infty$$

and, hence, $h_x \in W^{1,1}_{\text{loc}}$.

Exploring collisions: the collision set E_x

Proposition

The collision set $E_x = x^{-1}(O) \subset [0, T]$ is finite.

- Letting $I_x(t) := \frac{|x(t)|^2}{2}$, we have the virial identity:

$$I_x''(t) = \frac{1}{|x(t)|} + \langle \nabla_x U(t, x(t)), x(t) \rangle + 2h_x(t), \quad t \in [0, T] \setminus E_x.$$

- $I_x''(t) \to +\infty$ as t approaches a collision time, therefore $t \mapsto |x(t)|^2$ is strictly convex in a neighborhood of collision times.
- Collision times are isolated.

Exploring collisions: asymptotic directions at a collision time t_0

- For each small $\delta > 0$ there exist $t_\delta^-, t_\delta^+ > 0$ such that

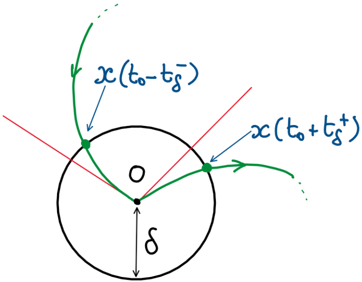
$$|x(t_0 \pm t_\delta^\pm)| = \delta$$
$$|x(t)| < \delta \quad \forall t \in \left]t_0 - t_\delta^-, t_0 + t_\delta^+\right[$$

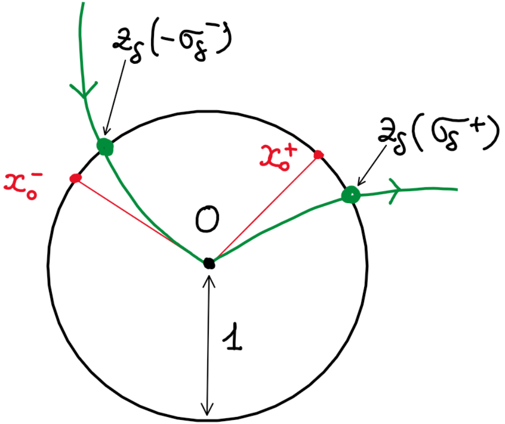
  and $t \mapsto |x(t)|^2$ is (strictly) convex in $[t_0 - t_\delta^-, t_0 + t_\delta^+]$.
- Sperling's asymptotics (Celestial Mech. 1969/70) at an isolated collision time t_0: there are two versors x_0^+ and x_0^- such that:

$$x(t) = \sqrt[3]{\frac{9}{2}}|t - t_0|^{2/3} x_0^\pm + o\left(|t - t_0|^{2/3}\right)$$
$$\dot{x}(t) = \frac{2}{3}\sqrt[3]{\frac{9}{2}}(t - t_0)^{-1/3} x_0^\pm + o\left(|t - t_0|^{-1/3}\right) \qquad \text{as } t \to t_0^\pm$$

Goal: $x_0^+ = x_0^-$.

Exploring collisions: blow-up analysis at t_0





Rescaling:
$z_\delta(s) := \frac{1}{\delta} x(\delta^{3/2} s + t_0)$ for $s \in [-\sigma_\delta^-, \sigma_\delta^+]$
$\sigma_\delta^\pm := t_\delta^\pm / \delta^{3/2}$ $|z_\delta(\sigma_\delta^\pm)| = 1$, $z_\delta(0) = O$,
$|z_\delta(t)| < 1 \ \forall t \in \left]-\sigma_\delta^-, \sigma_\delta^+\right[$.

Exploring collisions: blow-up analysis at t_0

$$z_\delta(s) := \frac{1}{\delta} x(\delta^{3/2} s + t_0), \quad s \in [-\sigma_\delta^-, \sigma_\delta^+] \qquad \left(\sigma_\delta^\pm := t_\delta^\pm / \delta^{3/2}\right)$$

$$|z_\delta(\sigma_\delta^\pm)| = 1, \qquad z_\delta(0) = O, \qquad |z_\delta(t)| < 1 \quad \forall t \in \left]-\sigma_\delta^-, \sigma_\delta^+\right[$$

A straightforward computation gives:

$$\frac{\mathcal{A}_{[t_0 - t_\delta^-, t_0 + t_\delta^+]}(x)}{\delta^{1/2}} = \int_{-\sigma_\delta^-}^{\sigma_\delta^+} \left(\frac{|\dot{z}_\delta|^2}{2} + \frac{1}{|z_\delta|} \right) + \delta^2 \int_{-\sigma_\delta^-}^{\sigma_\delta^+} U\left(t_0 + \delta^{3/2} s, z_\delta(s)\right) ds$$

Exploring collisions: blow-up analysis at t_0

Sperling's asymptotics as $\delta \to 0^+$ give that $\sigma_\delta^\pm \to s_0$, $z_\delta(s) \to \zeta(t; x_0^-, x_0^+)$ and $\dot{z}_\delta(s) \to \dot{\zeta}(t; x_0^-, x_0^+)$ $\forall 0 < |s| < s_0$, where:

$$\zeta(t; x_0^-, x_0^+) := \begin{cases} \sqrt[3]{\frac{9}{2}}|s|^{2/3} x_0^- & \text{if } -s_0 \le s \le 0, \\ \sqrt[3]{\frac{9}{2}}|s|^{2/3} x_0^+ & \text{if } 0 \le s \le s_0, \end{cases} \qquad (\text{b.t.w. } s_0 = \sqrt{2}/3).$$

is the parabolic collision-ejection solution of the following two-point bvp:

$$(2PK) \quad \begin{cases} \ddot{z} = -\frac{z}{|z|^3} & s \in [-s_0, s_0] \\ z(\pm s_0) = x_0^\pm \end{cases}$$

Moreover:

$$\liminf_{\delta \to 0^+} \frac{\mathcal{A}_{[t_0 - t_\delta^-, t_0 + t_\delta^+]}(x)}{\delta^{1/2}} \ge \psi_0 := \int_{-s_0}^{s_0} \left(\frac{|\dot{\zeta}|^2}{2} + \frac{1}{|\zeta|} \right) = 4\sqrt[3]{2\sqrt{2}}.$$

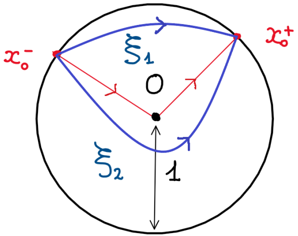
Exploring collisions: alternative routes

If $x_0^- \neq x_0^+$ it is known that $\zeta(\cdot; x_0^-, x_0^+)$ does not minimise the Keplerian action over the paths joining x_0^- to x_0^+ in the time interval $[-s_0, s_0]$.

Lemma [Fusco & Gronchi & Negrini, 2011]

If $x_0^- \neq x_0^+$ then there are exactly two classical solutions $\xi_i = \xi_i(\cdot; x_0^-, x_0^+)$ of $(2PK)$ (for $i = 1, 2$) such that:

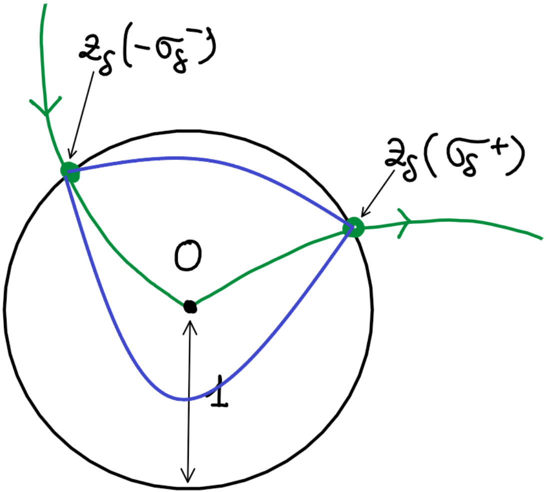
1. $\phi^i(x_0^-, x_0^+) := \int_{-s_0}^{s_0} \left(\frac{|\dot{\xi}_i|^2}{2} + \frac{1}{|\xi_i|} \right) < \psi_0$ for $i = 1, 2$;
2. they are not homotopic to each other in $\mathbb{R}^2 \setminus \{O\}$;
3. they depend smoothly on the data of the problem.

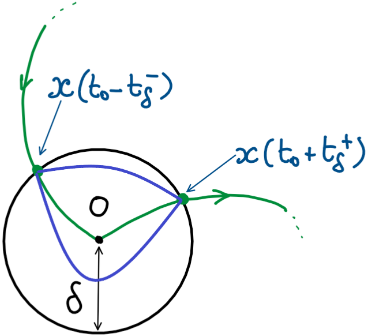


See also: Albouy, Lecture notes on the two-body problem (2002).

If we have $x_0^- \neq x_0^+$, we can use these ξ_i to modify x in a neighborhood of t_0 and decrease its action.

Exploring collisions: cut-and-paste near t_0





and x wouldn't anymore be minimal for $\mathcal{A}_T$ on $\mathcal{X}$.