Width parameters and graph classes: the case of mim-width

Flavia Bonomo-Braberman, Nick Brettell, Jake Horsfield, **Andrea Munaro**, Giacomo Paesani, and Daniël Paulusma

June 23, 2021

Queen's University Belfast

Motivation I

The following NP-hard problems are polynomial-time solvable on **triad-convex** graphs:

- DOMINATING SET (Pandey and Panda 2019)
- INDEPENDENT DOMINATING SET (Lu et al. 2013)
- CONNECTED DOMINATING SET (Liu et al. 2015)
- DOMINATING INDUCED MATCHING (Panda and Chaudhary 2019)
- FEEDBACK VERTEX SET (Jiang et al. 2013)

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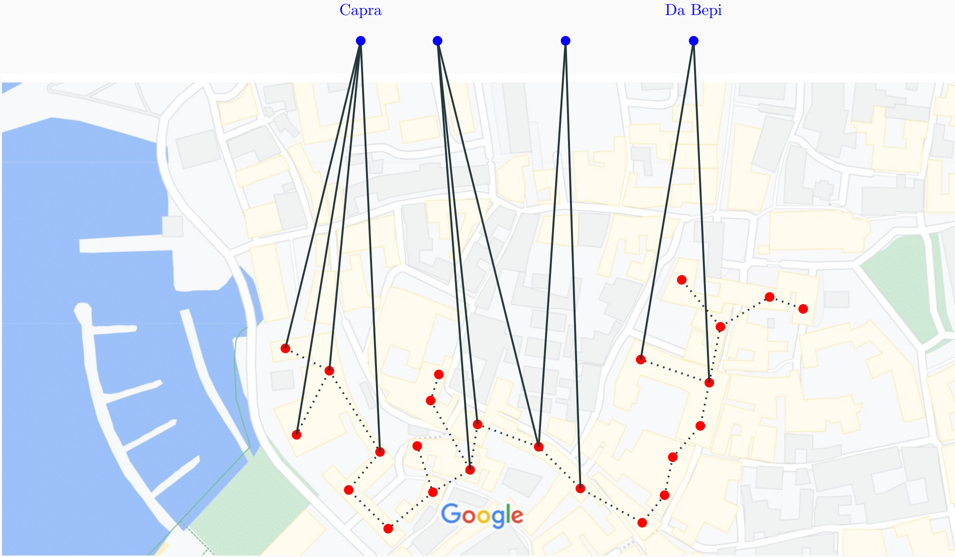
It turns out that:

- the problems above are polynomial-time solvable on graphs of bounded **mim-width**;
- triad-convex graphs have bounded **mim-width**.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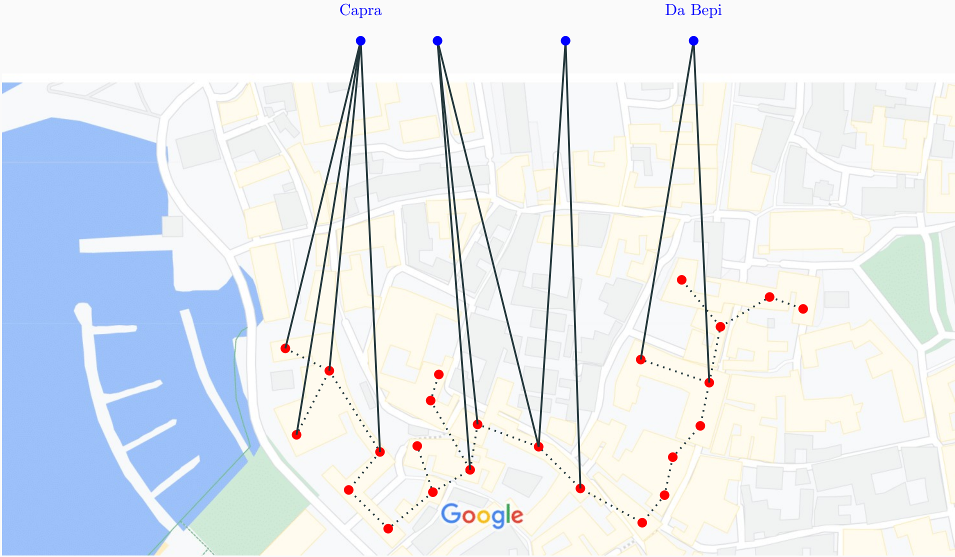
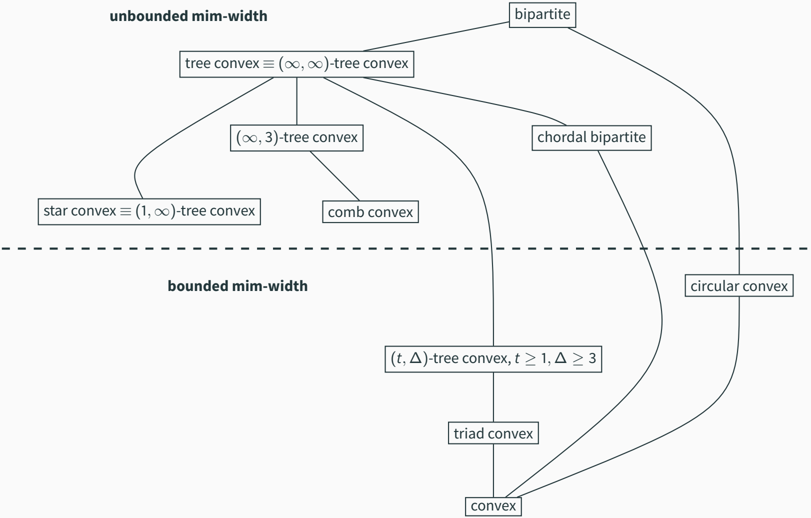
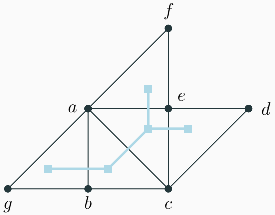


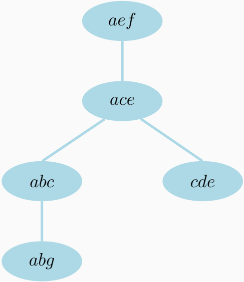
For $t, \Delta \geq 0$, a **(t, Δ)-tree** is a tree with maximum degree at most Δ and containing at most t vertices of degree at least 3.

Let $\mathcal{H}$ be a family of graphs. A bipartite graph $G = (A, B, E)$ is **$\mathcal{H}$-convex** if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the set of neighbours in A of each $b \in B$ induces a connected subgraph of H.



We all know why treewidth is useful





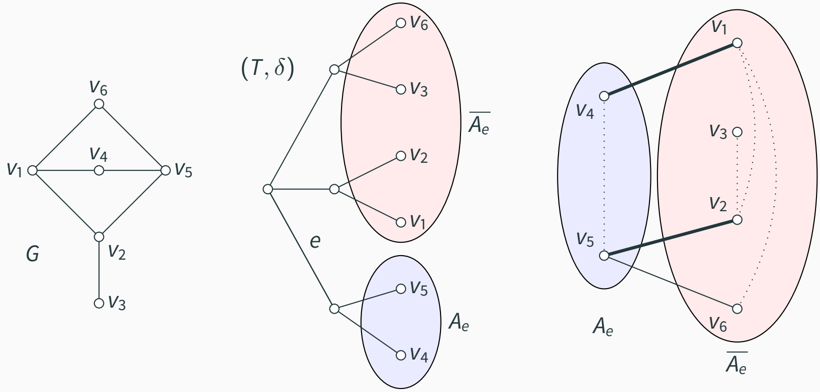
NP-hard to determine the treewidth of a graph (Arnborg et al. 1987).

$2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (Bodlaender 1996).

Dynamic Programming on tree decompositions.

Branch decompositions and mim-width

- Natural approach to DP: **recursively partition the vertices** of the graph into two parts.
- **Decomposition** of G can be stored **as a subcubic tree** whose leaves are in bijection with vertices of G.
- Need to store multiple sub-solutions at each intermediate node $\rightsquigarrow$ **structure of the cuts** is crucial to runtime.



Branch decomposition for G: (T, δ) where T is subcubic tree and δ is bijection between vertices of G and leaves of T. Each $e \in E(T)$ represents partition $(A_e, \overline{A_e})$ of $V(G)$.

$\mathrm{mimw}_G(T, \delta)$: $\max_{e \in E(T)}$ size of maximum induced matching in $G[A_e, \overline{A_e}]$.

$\mathrm{mimw}(G)$: min value of $\mathrm{mimw}_G(T, \delta)$ over all possible branch decompositions (T, δ) for G.

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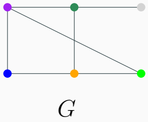

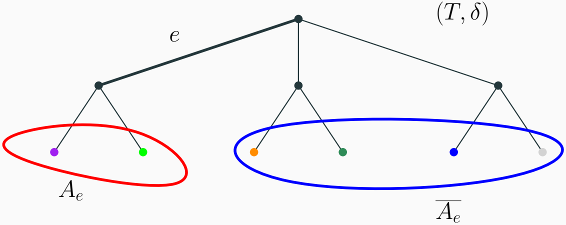

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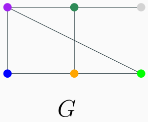

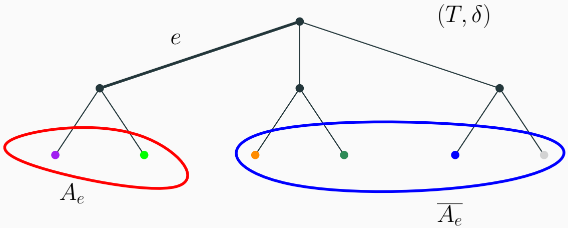

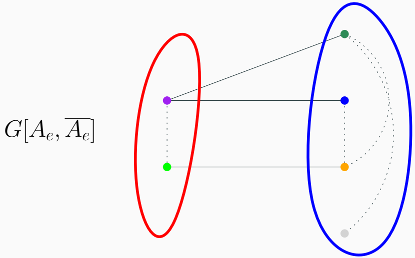

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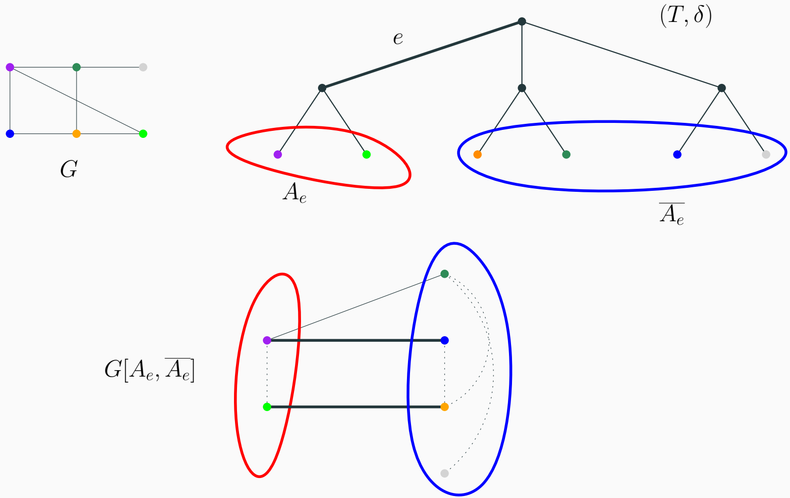
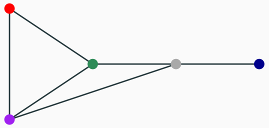


Interval graph: intersection graph of a family of intervals on the real line.



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Theorem (Belmonte, Vatshelle 2013)

$\mathrm{mimw}(G) \leq 1$, *for any interval graph G. Moreover, a branch decomposition of mim-width at most* 1 *can be computed in linear time.*

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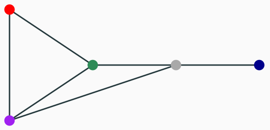

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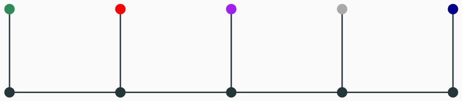
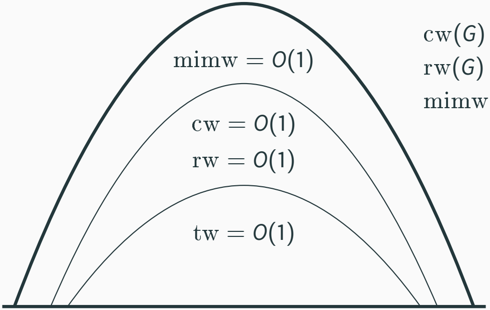


Why mim-width?



$\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)-1}$ (Corneil, Rotics 2005)

$\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1$ (Oum, Seymour 2006)

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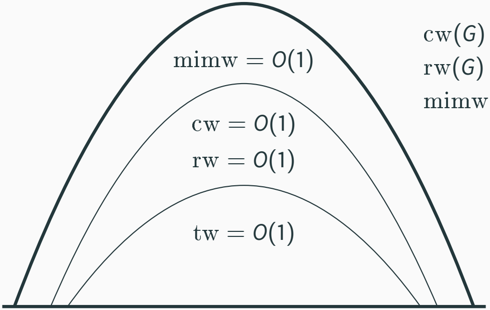

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- **LCVS problems in XP parameterized by mim-width, provided a branch decomposition is given** (Indep Set, Dom Set, Total Dom Set, Induced Matching, ...). Same for more general **LCVP problems** (k-Coloring, H-Homomorphism, ...).

(Bui-Xuan, Telle, Vatshelle 2013)

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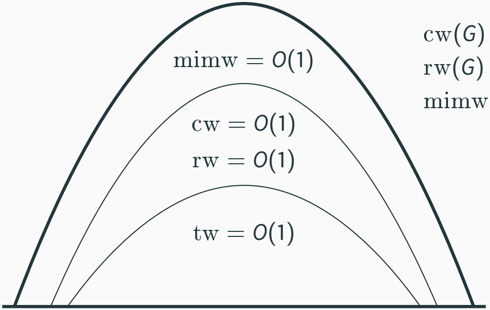

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- **Bad News:** In contrast to treewidth and rank-width, deciding mim-width is W[1]-hard.

  (Sæther and Vatshelle 2016)

LCVS problems

Given finite or co-finite subsets σ, ρ of $\mathbb{N}$ and a graph G, $S \subseteq V(G)$ is a **(σ, ρ)-set** if:

- $|N(v) \cap S| \in \sigma$, for each $v \in S$;
- $|N(v) \cap S| \in \rho$, for each $v \in V(G) \setminus S$.

Locally checkable vertex subset problem: find a min or max (σ, ρ)-set in input graph G (Telle and Proskurowski 1997).

Distance-r locally checkable vertex subset problem: replace $N(v)$ with $N^r(v)$ (Jaffke et al. 2020).

σ	ρ	d	Standard name
$\{0\}$	$\mathbb{N}$	1	Independent set $*$
$\mathbb{N}$	$\mathbb{N}^+$	1	Dominating set $**$
$\{0\}$	$\mathbb{N}^+$	1	Maximal Independent set $**$
$\mathbb{N}^+$	$\mathbb{N}^+$	1	Total Dominating set $\star\star$
$\{0\}$	$\{0,1\}$	2	Strong Stable set or 2-Packing
$\{0\}$	$\{1\}$	2	Perfect Code or Efficient Dom. set
$\{0,1\}$	$\{0,1\}$	2	Total Nearly Perfect set
$\{0,1\}$	$\{1\}$	2	Weakly Perfect Dominating set
$\{1\}$	$\{1\}$	2	Total Perfect Dominating set
$\{1\}$	$\mathbb{N}$	2	Induced Matching $\star$
$\{1\}$	$\mathbb{N}^+$	2	Dominating Induced Matching $\star$, $\star\star$
$\mathbb{N}$	$\{1\}$	2	Perfect Dominating set
$\mathbb{N}$	$\{d,d+1,...\}$	d	d-Dominating set $\star\star$
$\{d\}$	$\mathbb{N}$	$d+1$	Induced d-Regular Subgraph $\star$
$\{d,d+1,...\}$	$\mathbb{N}$	d	Subgraph of Min Degree $\geq d$
$\{0,1,...,d\}$	$\mathbb{N}$	$d+1$	Induced Subg. of Max Degree $\leq d$ $\star$

(Jaffke et al. 2020)

LCVS problems and mim-width

Theorem (Bui-Xuan et al. 2013)

*There is an algorithm that, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves each **LCVS** problem in $O(n^{4+3dw})$ **time**.*

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*There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves each **distance-r LCVS** problem in $O(n^{4+6dw})$ **time**.*

Solving distance-r LCVS on G is the same as solving distance-1 LCVS on G^r.
Moreover, $\mathrm{mimw}(G^r) \leq 2\mathrm{mimw}(G)$.

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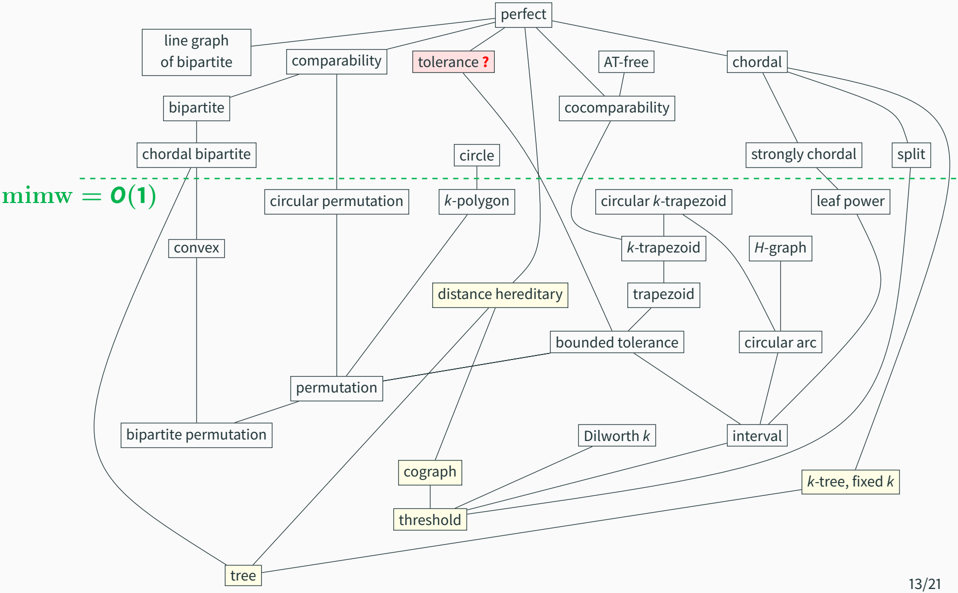
Theorem (Fomin et al. 2018)

Independent Set *and* Dominating Set *are W[1]-**hard parameterized by** $\mathrm{mimw}(G)$* ***and solution size****.*

Beyond LCVS problems

- Longest Induced Path (Jaffke et al. 2020)
- Induced Disjoint Paths (Jaffke et al. 2020)
- Feedback Vertex Set (Jaffke et al. 2020)
- Subset Feedback Vertex Set (Bergougnoux et al. 2020)
- Node Multiway Cut (Bergougnoux et al. 2020)
- Connected and acyclic variants of LCVS problems (Bergougnoux and Kanté 2019)
- Semitotal Dominating Set (Galby, M., Ries 2020)
- List k-Coloring (Kwon 2020)

Deciding mim-width is W[1]-hard: Not really a bad news



The case of k-Coloring

With mim-width we can again **simplify and generalize**:

Mim-width of $(K_r, K^1_{1,s}, P_t)$-free graphs is bounded and quickly computable

(Brettell, Horsfield, M., Paulusma 2020+)

- List k-Coloring in P for P_5-free graphs (Hoàng, Kamiński, Lozin, Sawada, Shu, 2010)
- List 3-Coloring in P for $(K^1_{1,s}, P_t)$-free graphs (Chudnovsky, Spirkl, Zhong 2020)
- Max Partial H-Coloring in P for P_5-free graphs with bounded clique number

  (Chudnovsky, King, Pilipczuk, Rzążewski, Spirkl 2020)
- Max Partial H-Coloring in P for $(K^1_{1,3}, P_6)$-free graphs with bounded clique number

  (Chudnovsky, King, Pilipczuk, Rzążewski, Spirkl 2020)

Mim-width classification for hereditary classes: motivation

Let $\mathcal{G}$ be a hereditary class with a forbidden set of induced subgraphs $\mathcal{F}$.

What can we say about (un)boundedness of mim-width for $\mathcal{G}$ when $|\mathcal{F}|$ is finite?

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What can we say about (un)boundedness of mim-width for $\mathcal{G}$ when $|\mathcal{F}|$ is finite?

- Analogous question for **tree-width well understood**

  (Bodlaender, Brettell, Johnson, Paesani, Paulusma, van Leeuwen 2020)

  (Lozin, Razgon 2020+)

- Analogous question for **clique-width/rank-width well understood when $|\mathcal{F}| \leq 2$**

  see (Dabrowski, Johnson, Paulusma 2019)

Our results

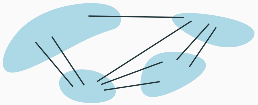
We obtain a partial picture when $|\mathcal{F}| \leq 2$ (Brettell, Horsfield, M., Paesani, Paulusma 2020)

Boundedness/unboundedness of mim-width resolved when:

- $|\mathcal{F}| = 1$
- $\mathcal{F} = \{H_1, H_2\}$ and H_1, H_2 are such that:
  - $|V(H_1)| + |V(H_2)| \leq 8$
  - **forests**, except for $H_1 = 2P_2$ and $H_2 \in \{K_{1,3} + sP_1, S_{1,1,2} + (s-1)P_1\}$ for $s \geq 1$
  - **connected**, except for:
    1. $H_1 = P_5$ and $H_2 = \overline{S_{1,1,2}}$ or $\overline{K_{1,r} + sP_1}$ for $r \geq 3$ and $s \in \{1, 2\}$
    2. $H_1 = P_7$ or $S_{h,i,j}$ for $h \leq i \leq j \leq 4$ with $i + j \leq 6 \leq h + i + j$ and $H_2 = C_3$ or paw
    3. $H_1 = K_{1,3}$ or $S_{1,1,2}$ and $H_2 =$ hammer

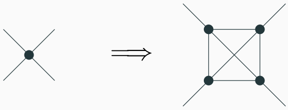
Complementation does not preserve mim-width.

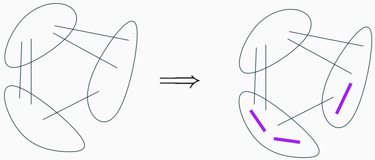
Upper bounds: **Ramsey-type arguments** and



Lower bounds:

- **Walls:** An $n \times n$ wall has mim-width at least $\sqrt{n}/50$.
- **1-subdivision of $e \in E(G)$:** $\mathrm{mimw}(G) \leq \mathrm{mimw}(G') \leq \mathrm{mimw}(G) + 1$.
- **Clique implant on $v \in V(G)$:** $\mathrm{mimw}(G) \leq \mathrm{mimw}(G') \leq \mathrm{mimw}(G) + d(v)$.
- **k-partite partial complementation:** $\mathrm{mimw}(G') \geq \mathrm{mimw}(G)/k$.
- **Blocks:** $\mathrm{mimw}(G) = \max\{\mathrm{mimw}(H) : H \text{ is a block of } G\}$.





Back to the beginning

Theorem (Bonomo-Braberman, Brettell, M., Paulusma 2021)

For $t, \Delta \in \mathbb{N}$, (t, Δ)-tree convex graphs can be recognized and a (t, Δ)-tree support computed, if it exists, in $O(n^{t+3})$ time.

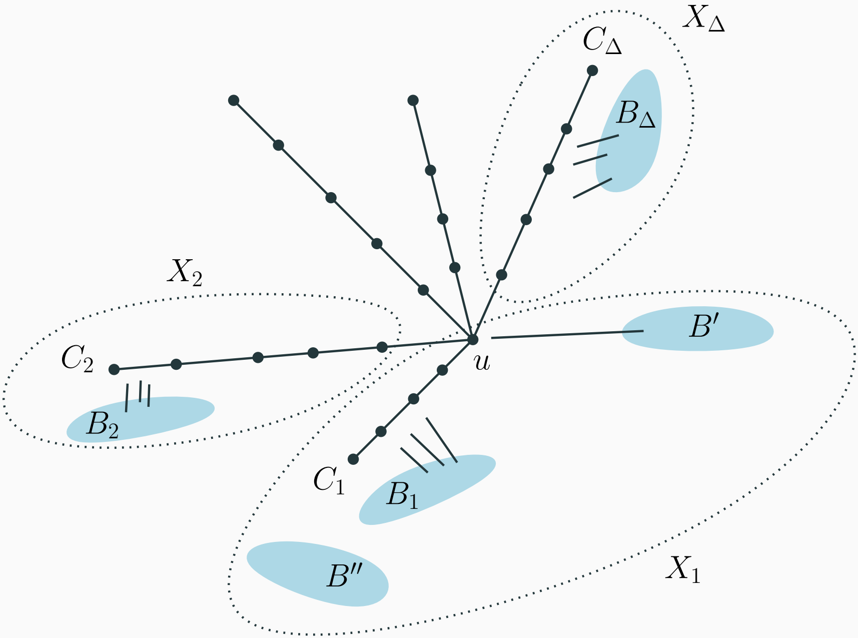
Let G be a (t, Δ)-tree convex graph with $t, \Delta \in \mathbb{N}$ and $t \geq 1$ and $\Delta \geq 3$. Then $\mathrm{mimw}(G) \leq f(t, \Delta)$. Moreover, we can construct in polynomial time a branch decomposition (T, δ) for G with $\mathrm{mimw}_G(T, \delta) \leq f(t, \Delta)$.

Lemma (Brettell, Horsfield, M., Paulusma 2020+)

Let G be a graph and $(X_1, \ldots, X_p)$ be a partition of $V(G)$ such that $\mathrm{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in \{1, \ldots, p\}$, and $p \geq 2$.

Let $h = \max\left\{c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor, \max_{i \in \{1,\ldots,p\}}\{\mathrm{mimw}(G[X_i])\} + c(p-1)\right\}$.

Then $\mathrm{mimw}(G) \leq h$. Moreover, given a branch decomposition (T_i, δ_i) for $G[X_i]$ for each i, we can construct in $O(p)$ time a branch decomposition (T, δ) for G with $\mathrm{mimw}_G(T, \delta) \leq h$.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C_Δ
B_Δ
X_2
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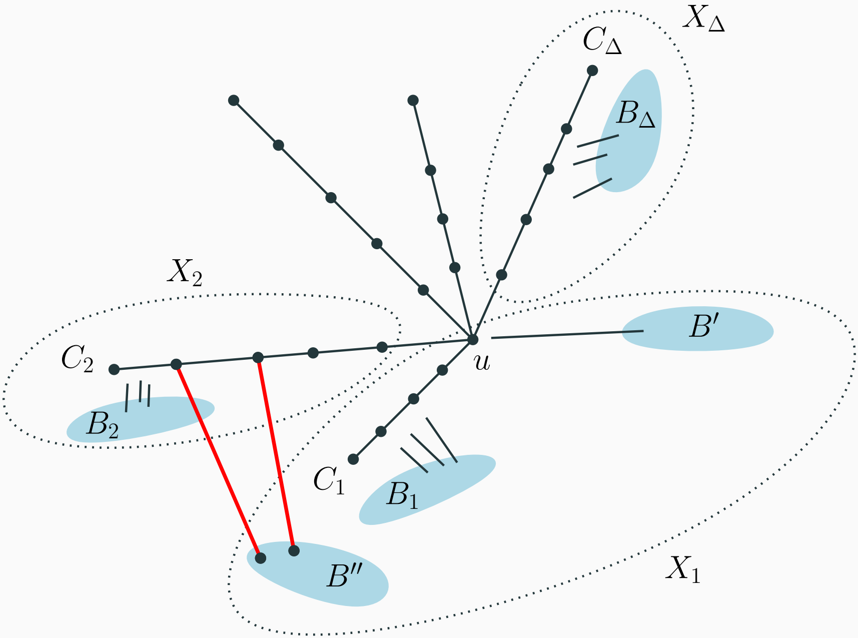
B'
C_1
B_1
B''
X_1



X_Δ
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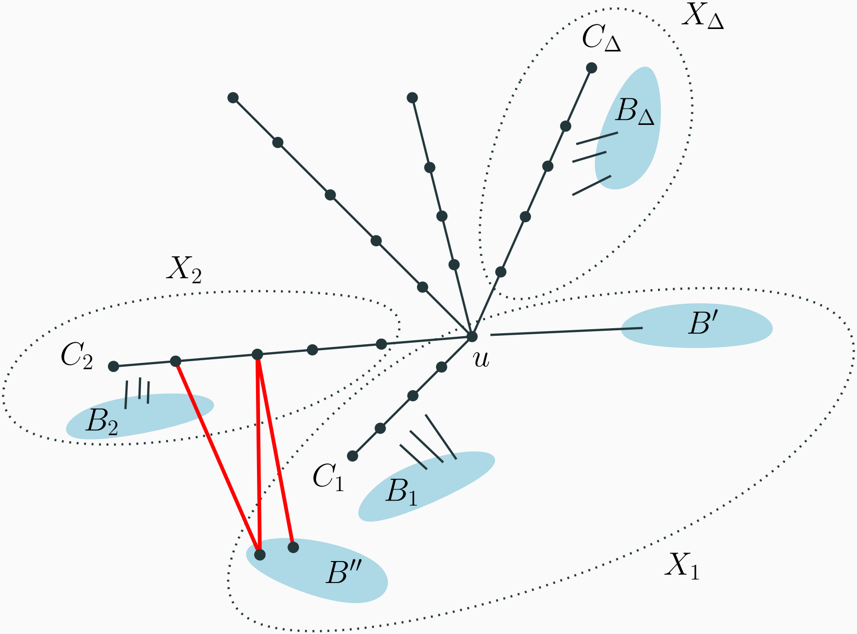

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B_2
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B_1
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X_1

GWP 2021

Registration

For free registration to the workshop, please fill in this form. The deadline for registration is Friday 2 July.

Format

This workshop will be held online, over Zoom.

It will be a one-day workshop, consisting of 6 invited talks, and finishing with a session for further discussion and open problems.

For the final session of the workshop, we invite short presentations that highlight an open problem or potential area for future research. If you wish to have a 10-minute slot i
description to a.munaro@qub.ac.uk by Friday 2 July 2021.

Speakers

- Eunjung Kim, LAMSADE, Paris-Dauphine University, France.
- Vadim Lozin, Mathematics Institute, University of Warwick, UK.
- Lalla Mouatadid, Department of Computer Science, University of Toronto, Canada.
- Paweł Rzążewski, Faculty of Mathematics and Information Science, Warsaw University of Technology, Poland.
- Jan Arne Telle, Department of Informatics, University of Bergen, Norway.
- David Wood, School of Mathematics, Monash University, Melbourne, Australia.

Programme

All times are in Central European Time.

9.30-10.15: David Wood – "The structure of planar graphs"
10.15-10.30: Break
10.30-11.15: Jan Arne Telle – "On Parameters in the Mim-width Family"
11.15-11.45: Break
11.45-12.30: Eunjung Kim – "Twin-width and Friends"
12.30-12.45: Break
12.45-13.30: Vadim Lozin – "A parametric approach to hereditary classes of graphs"
13.30-14.15: Break
14.15-15.00: Paweł Rzążewski – "The advantages of being modest: subexponential- and quasipolynomial-time algorithms for H-free graphs"
15.00-15.15: Break
15.15-16.00: Lalla Mouatadid – "Measuring Linear Structure on Graphs"
16.00-16.30: Break
16.30-18.30: Open problem session and discussion

Thank you!