Clique-Width: Harnessing the Power of Atoms

Konrad K. Dabrowski, Tomáš Masařík, Jana Novotná,
Daniël Paulusma, Paweł Rzążewski

University of Leeds, UK

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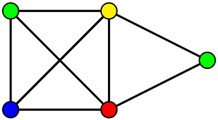
8ECM

Motivation

Consider an algorithmic problem e.g. Vertex colouring:



A proper colouring is an assignment of colours to the vertices of a graph such that no two adjacent vertices get the same colour.

A colouring using at most k colours is a k-colouring.

The Vertex Colouring Problem

Input: A graph G and an integer k
Question: Does G have a proper k-colouring?

This problem is NP-complete, even if $k = 3$.

What happens when the input is restricted to a class of graphs?

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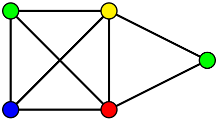

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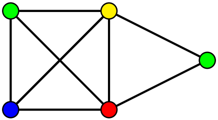

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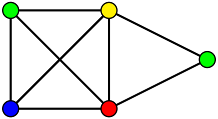

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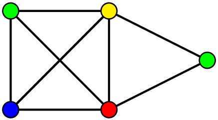

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Hereditary Classes

A graph H is an induced subgraph of G if H can be obtained by deleting vertices of G, written $H \subseteq_i G$.

P_4 $3P_1$ $P_1 + P_2$

So $P_1 + P_2 \subseteq_i P_4$, but $3P_1 \not\subseteq_i P_4$.

A class of graphs is hereditary if it is closed under taking induced subgraphs.

Let S be a set of graphs. The class of S-free graphs is the set of graphs that do not contain any graph in S as an induced subgraph.

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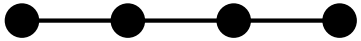

P_4

$3P_1$

$P_1 + P_2$

So $P_1 + P_2 \subseteq_i P_4$, but $3P_1 \not\subseteq_i P_4$.

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Colouring H-free graphs

Theorem (Král', Kratochvíl, Tuza & Woeginger, 2001)

The Vertex Colouring *problem is polynomial-time solvable for H-free graphs if and only if $H \subseteq_i P_1 + P_3$ or P_4, otherwise it is NP-complete.*

$P_1 + P_3$

P_4

Clique-width

Theorem (Courcelle, Makowsky & Rotics 2000, Kobler & Rotics 2003, Rao 2007, Oum 2008, Grohe & Schweitzer 2015)

Any problem expressible in "monadic second-order logic with quantification over vertices" (and certain other classes of problems) can be solved in polynomial time on graphs of bounded clique-width.

This includes:

- **Vertex Colouring**
- Maximum Independent Set
- Graph Isomorphism
- Minimum Dominating Set
- Hamilton Path/Cycle
- Partitioning into Perfect Graphs
- ...

Clique-width

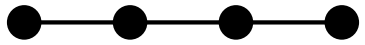
The clique-width is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i (represented by $i(v)$)

(ii) taking the disjoint union of two labelled graphs G_1 and G_2 (represented by $G_1 \oplus G_2$)

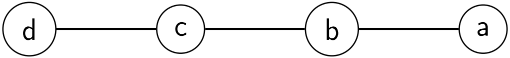
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$) (represented by $\eta_{i,j}$)

(iv) renaming label i to j (represented by $\rho_{i \to j}$)

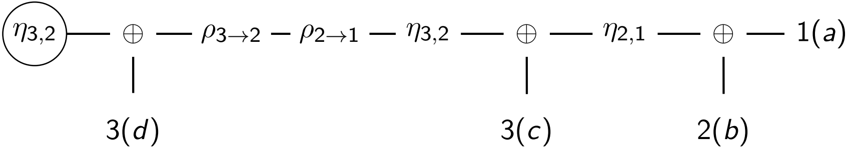


For example, P_4 has clique-width 3.

Clique-width



$$\eta_{3,2}(3(d) \oplus \rho_{3\to2}(\rho_{2\to1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))$$



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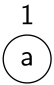


$1(a)$

$1(a)$

Clique-width

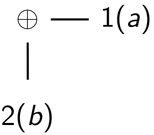
$2(b)$ $1(a)$

$1(a)$

$2(b)$

Clique-width

$$2(b) \oplus 1(a)$$



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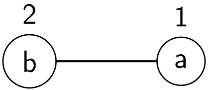

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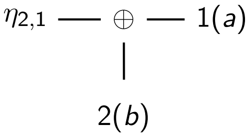

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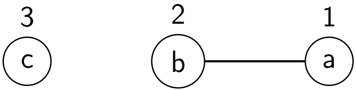

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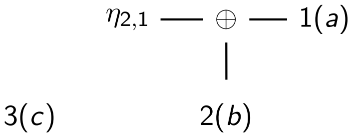

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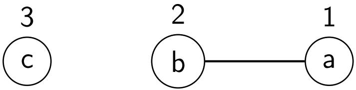

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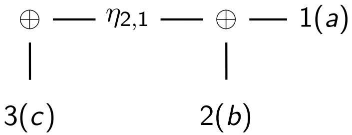
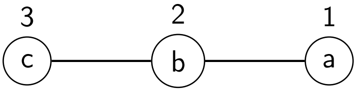
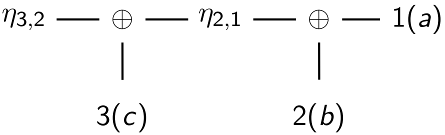


Clique-width



$$\textcolor{red}{\eta_{3,2}}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))$$



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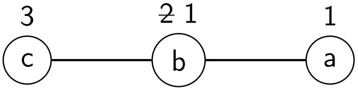

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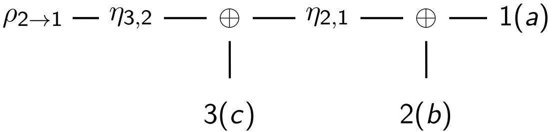

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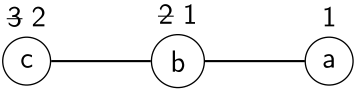

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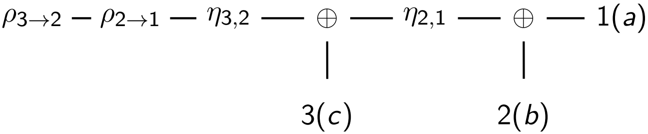

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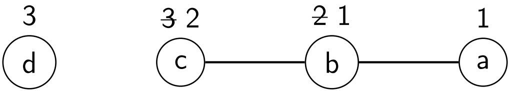

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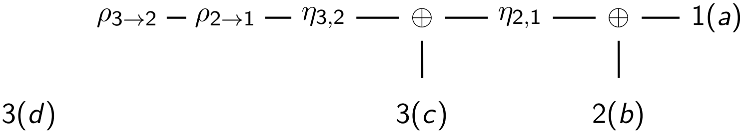

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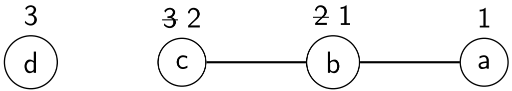

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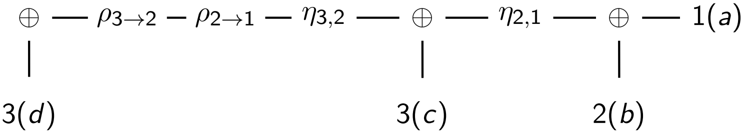

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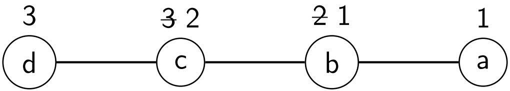

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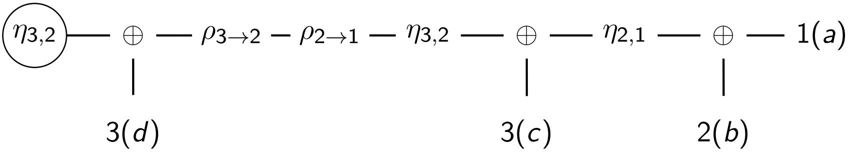

Why clique-width?

- "Equivalent" to rank-width and NLC-width
- Generalises tree-width
- "Equivalent" to tree-width on graphs of bounded degree

The following operations don't change the clique-width by "too much"

- Complementation
- Bipartite complementation
- Vertex deletion
- Edge subdivision (for graphs of bounded-degree)

Need only look at graphs that are

- prime
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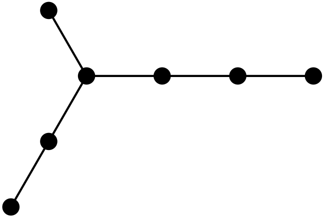
Which classes have bounded clique-width?

If the class of H-free graphs has bounded clique-width then every component of H must be a subdivided claw, path or isolated vertex. The set of such graphs is called $\mathcal{S}$.



$S_{1,2,3}$

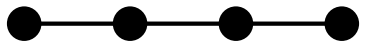
P_5

P_1

H-free graphs

Theorem (D., Paulusma 2015)

The class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$.



The classification of boundedness of clique-width on (H_1, H_2)-free graphs is known for all but five open cases.

Complexity of Vertex Colouring on (H_1, H_2)-free graphs is open for infinitely many cases.

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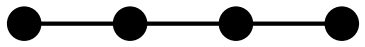

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Atoms

A graph is an atom if it has no clique cut-set.

For a hereditary class $\mathcal{H}$, we can solve the Vertex Colouring problem on graphs in $\mathcal{H}$ in polynomial time if we can do so for atoms in $\mathcal{H}$.

A vertex is simplicial if its neighbourhood is a clique.

If a graph is an atom, then it is either a clique, or it has no simplicial vertices.

For what classes of graphs does no simplicial vertex imply the graph is an atom?

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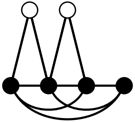
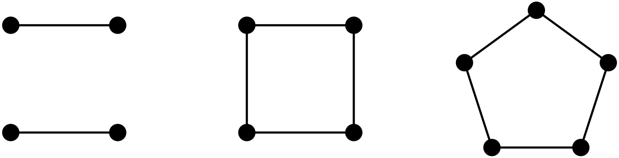
Split Graphs

A graph is split if its vertices can be partitioned into an independent set and a clique.



Equivalently, split graphs are the $(2P_2, C_4, C_5)$-free graphs.



$2P_2$ $\qquad$ C_4 $\qquad$ C_5

Every split graph has a simplicial vertex, so split atoms are cliques.

Also works for chordal graphs.

A class is nice if all connected graphs in it with no simplicial vertices are atoms.

Theorem (D., Paulusma 21+)

The class of H-free graphs is nice if and only if it is a subclass of: $2P_2$-free graphs or P_3-free graphs.

Theorem (D., Paulusma 21+)

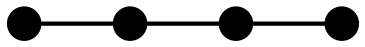
The class of (H_1, H_2)-free graphs is nice if and only if it is a subclass of:

- $2P_2$-free graphs
- P_3-free graphs
- $(P_1 + P_3, \overline{sunlet_4})$-free graphs
- $(P_4, \overline{P_1 + \overline{2C_4}})$-free graphs
- $(P_5, \overline{2P_1 + P_2})$-free graphs
- $(P_5, \overline{P_1 + P_3})$-free graphs
- $(C_4, 2P_3)$-free graphs or
- $(K_{1,3}, \overline{banner})$-free graphs

Are there classes which have unbounded clique-width, but whose atoms have bounded clique-width?

Theorem

The class of H-free atoms has bounded clique-width if and only if $H \subseteq_i P_4$.



- NO such cases for H-free graphs.
- YES: split graphs
- YES: chordal graphs
- YES: (cap, C_4)-free odd-signable graphs (Cameron, da Silva, Huang, Vušković, 2018)
- YES: (C_4, P_6)-free graphs (Gaspers, Huang, Paulusma, 2019)

What about (H_1, H_2)-free graphs?

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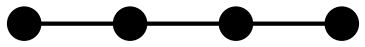

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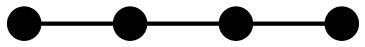

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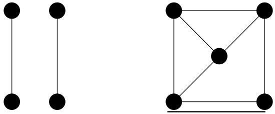
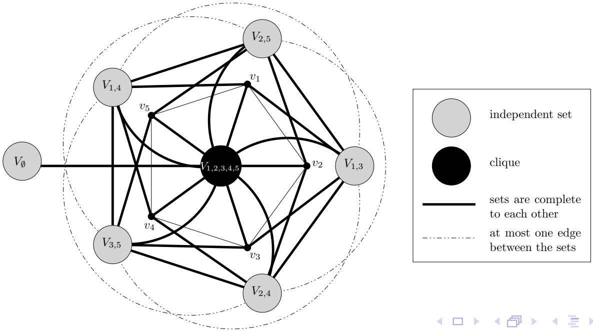
$(2P_2, \overline{P_2 + P_3})$-free Atoms Have Bounded Clique-width

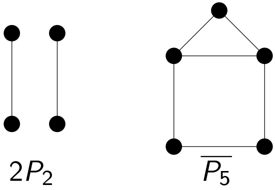


$2P_2$ $\quad$ $\overline{P_2 + P_3}$

- $(2P_2, \overline{P_2 + P_3})$-free graphs generalise split $((C_4, C_5, 2P_2)$-free$)$ graphs, so have unbounded clique-width
- $(2P_2, \overline{P_2 + P_3})$-free atoms containing an induced C_5
- $(C_5, 2P_2, \overline{P_2 + P_3})$-free atoms containing an induced C_4
- $(C_4, C_5, 2P_2, \overline{P_2 + P_3})$-free atoms are a subclass of split graphs



$(2P_2, \overline{P_5})$-free Atoms Have Unbounded Clique-width



- Take a split graph ($(2P_2, C_4, C_5)$-free, arbitrarily large clique-width)
- Add two non-adjacent vertices that are complete to the graph
- Result is a $(2P_2, \overline{P_5})$-free atom of arbitrarily large clique-width □

Open Problem

Does the class of (H_1, H_2)-free atoms have bounded clique-width if

(i) $H_1 = \text{diamond}$ *and* $H_2 = P_6$

(ii) $H_1 = C_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_4, 3P_2\}$

(iii) $H_1 = \overline{P_1 + 2P_2}$ *and* $H_2 \in \{2P_2, P_2 + P_3, P_5\}$

(iv) $H_1 = \overline{P_2 + P_3}$ *and* $H_2 \in \{P_2 + P_3, P_5\}$

*(v) $H_1 = K_3$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$

*(vi) $H_1 = 3P_1$ *and* $H_2 = \overline{P_1 + S_{1,1,3}}$

*(vii) $H_1 = \text{diamond}$ *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$

*(viii) $H_1 = 2P_1 + P_2$ *and* $H_2 \in \{\overline{P_1 + P_2 + P_3}, \overline{P_1 + P_5}\}$

*(ix) $H_1 = \text{gem}$ *and* $H_2 = P_2 + P_3$, *or*

*(x) $H_1 = P_1 + P_4$ *and* $H_2 = \overline{P_2 + P_3}$.

* means boundedness of clique-width is open for the whole class of (H_1, H_2)-free graphs

Summary

- Systematically studied boundedness of clique-width on (H_1, H_2)-free atoms
- 1 new bounded class
- Lots of unbounded classes
- There are 18* classes of (H_1, H_2)-free atoms for which boundedness of clique-width remains open
- There are 5* classes of (H_1, H_2)-free graphs for which boundedness of clique-width remains open

Further details: https://arxiv.org/abs/2006.03578

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Thank You!