# Optimization over trace polynomials

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Joint work with Igor Klep and Jurij Volcic

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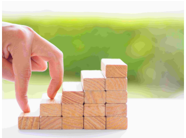

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sums of hermitian squares (SOHS): $f^\star f$ hermitian square
$S \subset \operatorname{Sym} \mathbb{T}$ $\quad X_j$ operators from finite von Neumann algebra
Constraints $\mathcal{D}_S = \{\underline{X} = (X_1, \ldots, X_n) : s(\underline{X}) \succcurlyeq 0, \quad \forall s \in S\}$

# Optimization over $\mathbb{T}$: special cases

- **Eigenvalue** optimization 💡 no traces [Helton-McCallough 04, Navascuez-Pironio-Acin 08]
  💡 Bounded self-adjoint operators on Hilbert spaces $\rightsquigarrow \mathcal{D}_S^\infty$

$$\lambda_{\min} = \inf \{ \langle a(\underline{X})\mathbf{v}, \mathbf{v} \rangle : \underline{X} \in \mathcal{D}_S^\infty, \|\mathbf{v}\| = 1 \}$$

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- Univ case [Klep-Pascoe-Volcic]: $a \succcurlyeq 0 \Rightarrow a =$ SOHS/SOHS
- Multilinear case [Huber]

# Motivation: quantum information

Entanglement in quantum mechanics
$\rightarrow$ **upper bounds** for violation levels of Bell inequalities

[Fukuda-Nechita 14] limit output states with input having specific parameters $\rightarrow$ bounds for generalized tensor traces

[Pozas et al 19] “scalar extension” of NPA hierarchy $\rightarrow$ identify correlations not attainable in entanglement-swapping scenario

NC operators $A_i, C_k$ satisfy causal constraints:

$$\operatorname{tr}(A_{i_1} \cdots A_{i_m} C_{k_1} \cdots C_{k_m}) - \operatorname{tr}(A_{i_1} \cdots A_{i_m}) \operatorname{tr}(C_{k_1} \cdots C_{k_m}) = 0.$$

Additional variables for each $\operatorname{tr}(w)$ but no convergence proof

# Contribution

**Theorem: T variant of Helton-McCullough Psatz**

Let $S \subset \operatorname{Sym} \mathbb{T}$ and $a \in$ T. The Positivstellensatz-induced hierarchy of semidefinite programs produces a convergent increasing sequence with limit $\inf_{\mathcal{D}_S^{\mathrm{II}_1}} a$.

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Also a cyclic Positivstellensatz for $\mathbb{T}$ (not enough time today)

Moment-sums of squares hierarchies

Non-cyclic Psatz for T

SDP hierarchies

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# Commutative Polynomial Optimization

**NP hard General Problem**: $f^* := \min_{\mathbf{x} \in \mathcal{D}_S} f(\mathbf{x})$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{s_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{s_2}$$

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Quadratic module: $\mathcal{M}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \ \deg \sigma_j \, s_j \leqslant 2r \right\}$

# Hierarchies for Polynomial Optimization

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Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

**"No Free Lunch" Rule**: $\binom{n+2r}{n}$ SDP variables

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$\mathcal{M}(S)$ **Archimedean quadratic module**: $N - \sum_i x_i^2 \succcurlyeq 0$

**Theorem: NC Putinar's Psatz [Helton-McCullough 02]**

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$a - \lambda \mathbf{1} = \sum_i h_i^\star h_i + \sum_j \sum_i t_{ji}^\star s_j t_{ji}$ with $h_i$, $t_{ji}$ of **bounded** degrees

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Disproving Connes' embedding conjecture: $\mathrm{tr}_{\min}^{\mathrm{II}_1} < \mathrm{tr}_{\min}$

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Converging hierarchy with cyclic quadratic modules:

💡 replace "$\mathrm{tr}(a - m) \geqslant 0$ on $\mathcal{D}_S^{\mathrm{II}_1}$" by $a - m\mathbf{1} \in \mathcal{M}^{\mathrm{cyc}}(S)_r$

$\mathcal{M}^{\mathrm{cyc}}(S)_r$ = polynomials with same trace as some from $\mathcal{M}(S)_r$

Moment-sums of squares hierarchies

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# Kadison-Dubois representation theorem

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**Theorem: Kadison-Dubois [Marshall 08]**

Given an Archimedean quadratic module $\mathcal{M} \subseteq \mathsf{T}$ & $a \in \mathsf{T}$:

$$\forall \varphi \in \chi_{\mathcal{M}} \quad \varphi(a) \geqslant 0 \qquad \Leftrightarrow \qquad \forall \varepsilon > 0 \quad a + \varepsilon \in \mathcal{M}$$

# Non-cyclic Psatz for T

For $S \subseteq \mathsf{T}$, "augment" $S$ with traces of hermitian squares:

$$S(N) = S \cup \{\mathrm{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \mathrm{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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**Proof**

By induction: $\forall w \in \langle \underline{x} \rangle$, $m \pm \operatorname{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\operatorname{tr}(x_j^k) = \left(N^k - \operatorname{tr}(x_j^{2k})\right) + \operatorname{tr}\left((x_j^k + 1)^2\right)$$

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$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = (N^k - \mathrm{tr}(x_j^{2k})) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

**Theorem: Non-cyclic Psatz for T [Klep-M.-Volcic 20]**

$a \geqslant 0$ on $\mathcal{D}_{S[N]}^{\mathrm{II}_1}$ $\quad\Leftrightarrow\quad$ $a + \varepsilon \in \mathcal{M}(S(N))$ for all $\varepsilon > 0$

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# Tracial words & moment matrices

$\mathbb{T}$-words = $\{\prod_i \operatorname{tr}(u_i)v \mid u_i, v \in \langle \underline{x} \rangle\}$ and T-words

$\operatorname{tr}(x_1)^2$ is a T-word, $\operatorname{tr}(x_1)x_1$ is a $\mathbb{T}$-word

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Reminder:

$$S(N) = S \cup \{\operatorname{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \operatorname{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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for $s \in S$, $a_i \in \mathsf{T}$, $f \in \mathbb{T}$

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There is no duality gap and $a_r \to a_{\min}^{\mathrm{II}_1}$ as $r \to \infty$

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💡 Reduction from the general trace setting to the pure trace

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# Linear Bell inequalities

CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leqslant 2$$

for separable states $\psi \in \mathbb{C}^k \otimes \mathbb{C}^k$ and matrices $A_j, B_j$ satisfying $A_j^* = A_j$, $A_j^2 = I$, $B_j^* = B_j$, $B_j^2 = I$

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# Polynomial Bell inequalities

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💡 2nd SDP relaxation of the corresponding trace problem outputs 5

$\Rightarrow$ The max value is 5 for **all** maximally entangled state

# Conclusion and perspectives

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# Thank you for your attention!

https://homepages.laas.fr/vmagron


- Klep & Magron & Volcic. Optimization over trace polynomials. arxiv:2006.12510
- Klep, Magron & Povh. Sparse Noncommutative Polynomial Optimization. Math Prog. A, arxiv:1909.00569


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