Groups acting with low fixity

ECM 2021, Minisymposium G2G2

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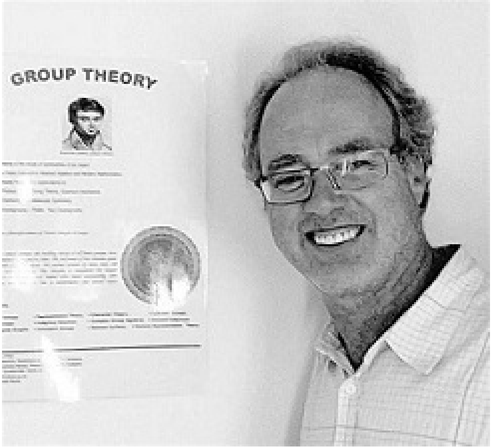

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"Have you seen this before?"

"I am pretty sure that we got the simple examples right. How would you prove that?"

How and why?

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How can we begin to understand the structure of G?

Setting the stage

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Fixity

Let $k \in \mathbb{N}$. We say that G **acts with fixity** k **on** Ω if and only if k is the maximal number of fixed points of elements of $G^{\#}$. (Ronse 1980)

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3. The simple groups that Kay mentioned are indeed examples for fixity 2, and there are infinite families of groups that act with fixity 3 and 4 as well.

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In the remainder of the talk, I will refer to this part of the project by **"classification problem"**.

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Classify the possibilities where

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We also prove structure results for general finite groups that act with fixity 2, and a revision of this work is in progress (with Anika Streck).

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If q is a prime power and G is isomorphic to $\mathrm{PSL}_2(q)$ or to $\mathrm{Sz}(q)$, or if $G \cong \mathrm{PSL}_3(4)$, then the realisation problem is solved. Specifications are given by ramification data.

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How can we use the fixity hypothesis and find information about the structure of G?

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As a direct consequence, we can bound $|Z(G)|$ and analyse the connection between Sylow subgroups and point stabilisers.

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This has consequences for the possible orbit lengths of Sylow subgroups and for the structure of $F^*(G)$, and it gives rise to a strategy for solving the classification problem.

Strategy for fixity 4, simple groups

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- The realisation problem for groups of fixity 4. (Salfeld, W.)

Thanks for listening!

I look forward to your comments and questions.