On some fractional problems with Dirichlet-Neumann boundary conditions

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The main results of the talk are collected in:

E. C., A. Ortega, *The Brezis-Nirenberg problem for the fractional Laplacian with mixed Dirichlet-Neumann boundary conditions. J. Math. Anal. Appl.* ***473*** *(2019).*

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P. Álvarez-Caudevilla, E. C., A. Ortega, *The Positive solutions for semilinear elliptic problems involving an inverse fractional operator. Nonlinear Anal. Real World Appl.* ***51*** *(2020).*

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- Definition of the Fractional Laplacian (through the spectral decomposition) and extended problem to one more variable

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- Attainability of the Sobolev constant
- Brezis-Nirenberg fractional problem with mixed Dirichlet-Neumann (D-N) boundary conditions
- Fractional elliptic problems involving an inverse fractional operator

Fractional Laplacian with D-N boundary data

Powers of Laplacian operator $(-\Delta)$:

Let (λ_n, φ_n) be the eigenvalues and eigenfunctions of $(-\Delta)$ in Ω with zero mixed D-N boundary data. Then (λ_n^s, φ_n) are the eigenvalues and eigenfunctions of $(-\Delta)^s$, also with zero D-N boundary conditions.

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The fractional Laplacian $(-\Delta)^s$ is well defined in the space of functions that vanish on $\Sigma_{\mathcal{D}}$,

$$H^s_{\Sigma_{\mathcal{D}}}(\Omega) = \left\{ u = \sum_{n\geq 1} a_n \varphi_n \in L^2(\Omega) : \ \|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \sum_{n\geq 1} a_n^2 \lambda_n^s < \infty \right\}.$$

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As a consequence,

$$(-\Delta)^s u = \sum_{n\geq 1} \lambda_n^s a_n \varphi_n .$$

Note that then $\|u\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}$.

Fractional Laplacian with D-N boundary data

Following **[LM]**

- $H_0^s(\Omega) = H^s(\Omega)$ for $0 < s \leq \frac{1}{2}$.
- $H_0^s(\Omega) \subsetneq H^s(\Omega)$ for $\frac{1}{2} < s < 1$.

[LM] J.-L. Lions, E. Magenes, *A Non-homogeneous boundary value problems and applications. Vol. I, Springer-Verlag, New York-Heidelberg, 1972.*

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As a consequence

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Fractional Laplacian with D-N boundary data

For the general problem

$$(P) \quad \begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where we take mixed Dirichlet-Neumann boundary conditions,

$$B(u) = \chi_{\Sigma_{\mathcal{D}}}\, u + \chi_{\Sigma_{\mathcal{N}}}\, \frac{\partial u}{\partial \nu}.$$

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$$B(u) = \chi_{\Sigma_{\mathcal{D}}} u + \chi_{\Sigma_{\mathcal{N}}} \frac{\partial u}{\partial \nu}.$$

- $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$.
- $\Sigma_{\mathcal{D}}$ is a closed manifold of positive $(N-1)$-dimensional Hausdorff measure,

$$\mathcal{H}^{N-1}(\Sigma_{\mathcal{D}}) = \alpha \in (0, \mathcal{H}^{N-1}(\partial\Omega)).$$

- $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ verify $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$, $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$, where Γ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.

Fractional Laplacian with D-N boundary data

$$
(P_\lambda) \quad \begin{cases} (-\Delta)^s u = \lambda u + u^{\frac{N+2s}{N-2s}}, \quad u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\lambda > 0$, and $\Omega \subset \mathbb{R}^N$, with $N > 2s$, $\frac{1}{2} < s < 1$.

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Sense of weak/energy solution

$$\int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx = \int_\Omega f_\lambda(u) \varphi \, dx, \quad \forall \varphi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

We also have an associated energy functional ($2_s^* = \frac{2N}{N-2s}$)

$$I(u) = \frac{1}{2} \int_\Omega \left| (-\Delta)^{s/2} u \right|^2 dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{2_s^*} \int_\Omega u^{2_s^*} dx$$

which is well defined in $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$. Clearly, the critical points of I correspond to solutions to (P_λ).

Extended problems to one more variable

Consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. Given $u \in H^s_{\Sigma_\mathcal{D}}(\Omega)$, we define its s-harmonic extension $w = \mathrm{E}_s(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ B^*(w) = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times [0,\infty), \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

where

$$B^*(w) = w\chi_{\Sigma^*_\mathcal{D}} + \frac{\partial w}{\partial \nu}\chi_{\Sigma^*_\mathcal{N}},$$

with $\Sigma^*_\mathcal{D} = \Sigma_\mathcal{D} \times [0,\infty)$ and $\Sigma^*_\mathcal{N} = \Sigma_\mathcal{N} \times [0,\infty)$.

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with $\Sigma^*_\mathcal{D} = \Sigma_\mathcal{D} \times [0,\infty)$ and $\Sigma^*_\mathcal{N} = \Sigma_\mathcal{N} \times [0,\infty)$.

The extension function belongs to the space $X^s_{\Sigma^*_\mathcal{D}}(\mathcal{C}_\Omega)$ defined as the completion of $\{z \in \mathcal{C}^\infty(\mathcal{C}_\Omega) : \ z = 0 \text{ on } \Sigma^*_\mathcal{D}\}$ with respect to the norm

$$\|z\|_{X^s_{\Sigma^*_\mathcal{D}}(\mathcal{C}_\Omega)} = \left(\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z|^2 dxdy\right)^{1/2}$$

where κ_s is a normalization constant.

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with $\Sigma^*_\mathcal{D} = \Sigma_\mathcal{D} \times [0,\infty)$ and $\Sigma^*_\mathcal{N} = \Sigma_\mathcal{N} \times [0,\infty)$.

Note that the extension operator is an isometry

$$\|\mathsf{E}_s(\psi)\|_{X^s_{\Sigma^*_\mathcal{D}}(\mathcal{C}_\Omega)} = \|\psi\|_{H^s_{\Sigma_\mathcal{D}}(\Omega)}, \quad \forall\, \psi \in H^s_{\Sigma_\mathcal{D}}(\Omega).$$

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Moreover, for any $\varphi \in X^s_{\Sigma^*_\mathcal{D}}(\mathcal{C}_\Omega)$, we have the following trace inequality

$$\|\varphi\|_{X^s_{\Sigma^*_\mathcal{D}}(\mathcal{C}_\Omega)} \geq \|\varphi(\cdot,0)\|_{H^s_{\Sigma_\mathcal{D}}(\Omega)}.$$

Extended problems to one more variable

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$-\kappa_s \lim_{y\searrow 0} y^{1-2s}\frac{\partial w}{\partial y}(x,y) = (-\Delta)^s u(x),$$

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[CS] L. Caffarelli, L. Silvestre, CPDE, 2007.

See also:

[BCdPS] C. Brändle, E.C., A. de Pablo, U. Sánchez, PRSE, 2013.

[CT] X. Cabré, J. Tan, Adv. Math., 2010.

[CDDS] A. Capella, J. Dávila, L. Dupaigne, Y. Sire, CPDE, 2011.

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Denoting

$$L_s w := -\mathrm{div}(y^{1-2s}\nabla w), \qquad \frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y\searrow 0} y^{1-2s}\frac{\partial w}{\partial y}$$

we can reformulate (P_λ) with the new variable as

$$(P_\lambda^*) \quad \begin{cases} L_s w = 0 & \text{in } \mathcal{C}_\Omega, \\ B^*(w) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial w}{\partial \nu^s} = \lambda w + w^{\frac{N+2s}{N-2s}} & \text{in } \Omega \times \{y = 0\}. \end{cases}$$

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We say as before that $w \in X^s_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega)$ is an energy solution if

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Energy functional

$$J(w) = \frac{\kappa_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2\, dxdy - \frac{\lambda}{2}\int_\Omega w^2\, dx - \frac{1}{2_s^*}\int_\Omega w^{2_s^*}\, dx\,.$$

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Note that critical points of J in $X^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ correspond to critical points of I in $H^s_{\Sigma^*_{\mathcal{D}}}(\Omega)$.

Sobolev and Trace inequalities (Mixed D-N)

Since we have a Dirichlet condition on $\Sigma_{\mathcal{D}}$ with $0 < \mathcal{H}^{N-1}(\Sigma_{\mathcal{D}}) < \mathcal{H}^{N-1}(\partial\Omega)$, then

$$0 < C := \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|_{L^{2^*_s}(\Omega)}}.$$

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Hence, in terms of the extension function,

$$\left(\int_\Omega \varphi^{\frac{2N}{N-2s}}(x,0)dx\right)^{\frac{N-2s}{2N}} \leq C\|\varphi(\cdot,0)\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = C\|E_s[\varphi(\cdot,0)]\|_{X^s_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega)}.$$

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As a consequence, we obtain the following **Mixed Trace inequality**,

$$\left(\int_\Omega \varphi^{\frac{2N}{N-2s}}(x,0)dx\right)^{1-\frac{2s}{N}} \leq C\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla\varphi|^2 dxdy.$$

for any $\varphi \in X^s_{\Sigma^*_{\mathcal{D}}}(\mathcal{C}_\Omega)$, where C is a positive constant.

Sobolev constant relative to $\Sigma_{\mathcal{D}}$

We define the Sobolev constant **"relative to $\Sigma_{\mathcal{D}}$"** as follows,

$$S(\Sigma_{\mathcal{D}}) = \inf_{\substack{u\in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\\ u\not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}} = \inf_{\substack{w\in X^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{X^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}}.$$

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Theorem 1. $S(\Sigma_{\mathcal{D}}) \leq 2^{-\frac{2s}{N}}\kappa_s S(s,N)$, and even more, if $S(\Sigma_{\mathcal{D}}) < 2^{-\frac{2s}{N}}\kappa_s S(s,N)$ $\Rightarrow S(\Sigma_{\mathcal{D}})$ is attained.

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The key of the proof relies on concentration-compactness arguments by Lions **[L]**. See **[ACP]** for similar arguments adapted to mixed problems with $s = 1$.

[ACP] A. Abdellaoui, E.C., I. Peral, ADE, 2004.

[L] P.L. Lions, Rev.Mat.Iber, 1985.

Sobolev constant relative to $\Sigma_{\mathcal{D}}$

Following **[CP, Lemma 4.1]** we have the next result.

Lemma 1. Under certain geometrical assumptions on the distribution of $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ on $\partial\Omega$, $\lambda_1^s(\alpha) \to 0$, as $\alpha = \mathcal{H}^{N-1}(\Sigma_{\mathcal{D}}) \to 0$.

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The proof follows by using Theorem 1 and Lemmas 1-2 jointly because $S(\Sigma_{\mathcal{D}})$ is as small as we want provided $\alpha \to 0$, proving that $S(\Sigma_{\mathcal{D}}) < 2^{-\frac{2s}{N}}\kappa_s S(s,N)$.

Main Results

Remember the main problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^s u = \lambda u + u^{\frac{N+2s}{N-2s}}, \quad u > 0 & \text{in } \Omega, \\ B(u) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, and $\Omega \subset \mathbb{R}^N$, with $N > 2s$, $\frac{1}{2} < s < 1$.

Main Results

Theorem 3. Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Then problem (P_λ):

1. has no solution for $\lambda \geq \lambda_1^s$,
2. has solution for each $0 < \lambda < \lambda_1^s$,
3. under the some geometrical assumptions, has solution for $\lambda = 0$ and $\mathcal{H}^{N-1}(\Sigma_{\mathcal{D}})$ sufficiently small.

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Variational approach: minimizers

To prove point 2 in Theorem 3, i.e., the existence of solution to (P_λ), for $0<\lambda<\lambda_1^s$, we consider the following quotient

$$Q_\lambda(w)=\frac{\|w\|^2_{X^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}-\lambda\|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}},$$

where $w=E_s[u]$, and we define

$$S_\lambda(\Omega)=\inf_{\substack{w\in X^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)\\ w\not\equiv 0}}\{Q_\lambda(w)\},$$

in order to find a minimizer.

Fractional elliptic problems, inverse fractional operator

$$(P_{\alpha,\beta}) \begin{cases} (-\Delta)^{\alpha-\beta}u = \lambda(-\Delta)^{-\beta}u + |u|^{2^*_\mu-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

We prove existence or nonexistence of positive solutions depending on the parameter $\lambda > 0$, up to the critical value of the exponent p, i.e., for $1 < p \le 2^*_\mu - 1$ where $\mu := \alpha - \beta$ and $2^*_\mu = \frac{2N}{N-2\mu}$ is the critical exponent of the Sobolev embedding.

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Theorem. For every $\gamma \in (0, \lambda_1^\alpha)$, there exists a positive solution for the problem $(P_{\alpha,\beta})$ provided that $N > 4\alpha - 2\beta$.

Thank you for the attention!