Embedding spheres in knot traces

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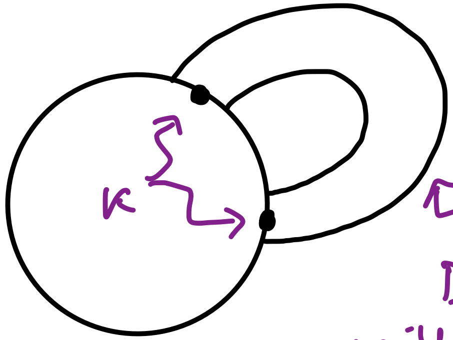
Question: Given M^4 and $x \in \pi_2 M$,
is x represented by a locally flat, embedded sphere?

$X_n(K) :=$ 

$D^2 \times D^2$ with framing $n \in \mathbb{Z}$

$\simeq S^2$

$\pi_2(X_n(K)) \cong \mathbb{Z}$

Theorem [FMNOPR]: Fix $n \in \mathbb{Z}$, $K \subseteq S^3$ a knot.

A generator of $\pi_2(X_n(K))$ is represented by a loc. flat emb S with $\pi_1(X_n(K) \setminus S)$ abelian

if and only if

(i) $\mathrm{Arf}(K) = 0$, (ii) $H_1(\widetilde{S^3_n(K)}; \mathbb{Z}) = 0$, (iii) $\sigma_K(\xi) = 0 \;\forall \xi$ with $\xi^n = 1$.

Cases:

- For $n = \pm 1$, gen of $\pi_2(X_{\pm 1}(K))$ is rep by a loc. flat emb

iff

$$\mathrm{Arf}(K) = 0$$

- For $n = 0$, gen of $\pi_2(X_0(K))$ is rep by a loc. flat emb S with $\pi_1(X_0(K) \setminus S) \cong \mathbb{Z}$

iff

$$\Delta_K(t) \doteq 1$$

- For $n \neq 0$, gen of $\pi_2(X_n(K))$ is rep by a loc. flat emb S with $\pi_1(X_n(K) \setminus S) \cong \mathbb{Z}/n$

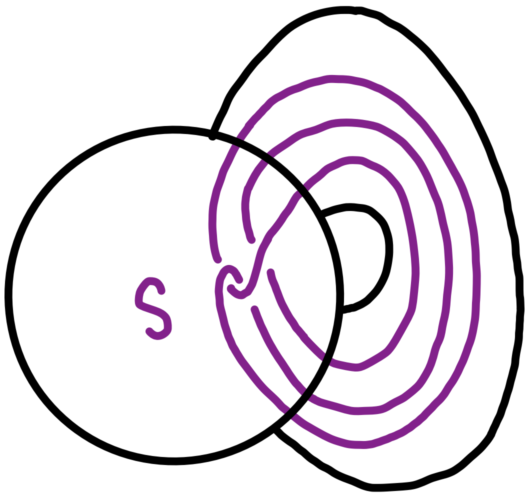
iff

(i) $\mathrm{Arf}(K) = 0$, (ii) $\prod_{\xi^n = 1} \Delta_K(\xi) = 1$, (iii) $\sigma_K(\xi) = 0 \ \forall \xi$ with $\xi^n = 1$.

Proof sketch: Fix n, K.

Suppose $\exists S$ emb, rep gen of $\pi_2(X_n(K))$ with $\pi_1(X_n(K)\setminus S)$ abelian.

$Q_{X_n(K)} = \langle n \rangle$



$\partial X_n(K) = S^3_n(K)$

$\nu S \cong D_n :=$ D^2 bundle over S^2 with euler number n.

$\partial D_n = L(n,1)$ lens space.

$V := X_n(K) \setminus \nu S$ is a hom. cob. between $S^3_n(K)$ and $L(n,1)$

$\pi_1 V \cong \mathbb{Z}/n$

$V \cup D_n \cong X_n(K)$.

$(\Longleftarrow)$

Step 1: Build a TOP hom cob. V between $S^3_n(K)$ and $L(n,1)$

with $\pi_1 V \cong \mathbb{Z}/n$ Idea: use (TOP) surgery.

Input to surgery is $\tilde{V} \longrightarrow L(n,1) \times [0,1]$ "degree one normal map"

NEED: n odd ✓ n even: $\mathrm{Arf}(K)=0$

The H_* condition $\Longrightarrow$ surgery obstr. lies in $L^s_4(\mathbb{Z}[\mathbb{Z}/n])$

computed in terms of "multisignature" $\to$ Levine-Tristram signature

$\partial V = S^3_n(K) \sqcup L(n,1)$

Step 2: Show $X := V \cup D_n \overset{\text{homeo}}{\cong} X_n(K)$.

D_n: embedded S^2 rep gen of π_2

Boyer 1986: classified simply connected 4-mflds w. ∂.

$\pi_1 X = 1$, $\pi_2 \cong \mathbb{Z}$, $Q_X = \langle n \rangle$

Is $X \cong X_n(K)$?

n even ✓

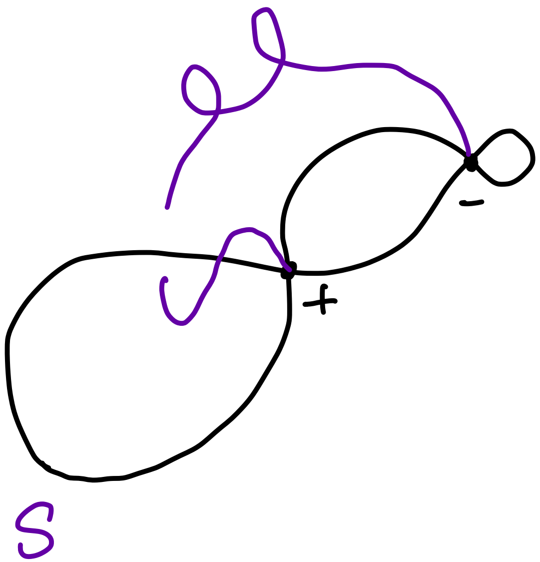
n odd iff $\mathrm{Arf}(K)=0$.

$$\boxed{\begin{array}{l}\text{Goal:}\\ ks(X)=0 \\ \iff X \cong X_n(K)\end{array}}$$

Goal: n odd, $X := V \cup D_n$, $ks(X) = \mathrm{Arf}(K)$.

Tool: An invariant of immersed spheres.

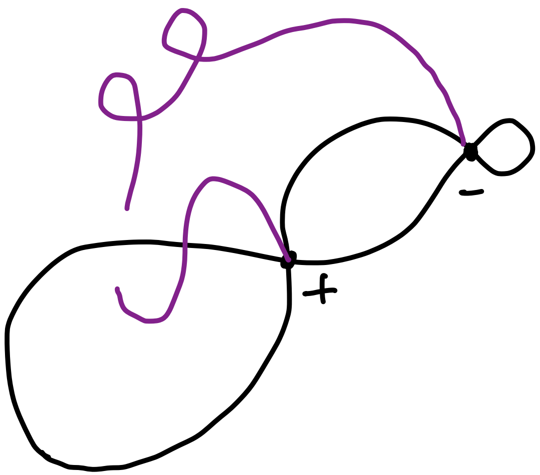
Let M^4 be simply connected. $S^2 \looparrowright M$ immersed.



Assume $\mu(S) = 0$

$\exists \{W_i\}$ (framed) immersed whitney discs pairing up the intersections.

$$\tau(S; \{W_i\}) := \sum_i |S \pitchfork \mathring{W}_i| \mod 2$$



$$\tau(S;\{W_i\}) := \sum_i |S \pitchfork \mathring{W}_i| \bmod 2$$

[Schneiderman-Teichner] $\tau(S;\{W_i\})$ is independent of $\{W_i\}$

iff

S is s-characteristic

i.e. $S \pitchfork S \equiv R \pitchfork R^{+}$ $\forall$ immersed spheres $R \subseteq M$

τ is an invariant of s-characteristic homotopy classes.

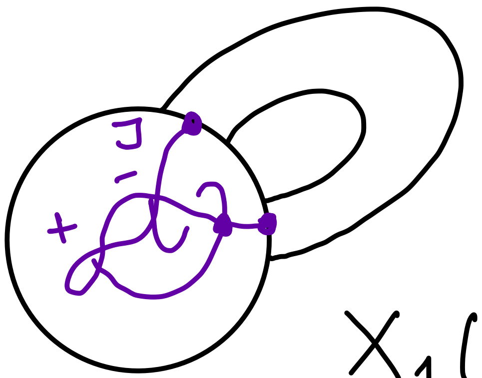
Using τ to distinguish between CP^2 and $*CP^2$

$$CP^2 = X_1(U) \cup B^4$$

Chern manifold $*CP^2 := X_1(J) \cup -C$

$\text{Arf}(J) = 1$

contractible 4-mfld [Freedman]



$J = RHT$

$X_1(J)$

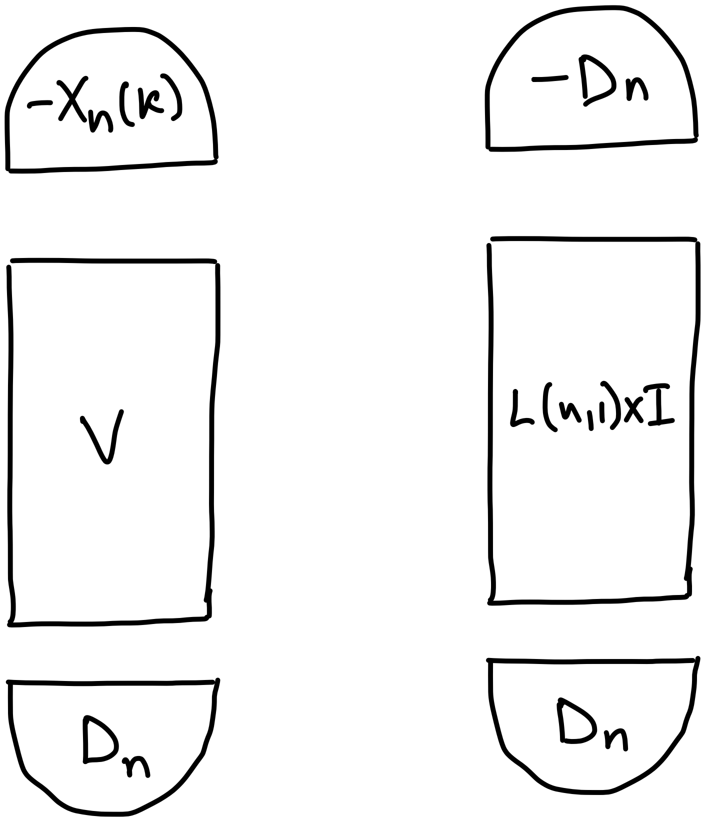
$$*CP^2 \simeq CP^2$$

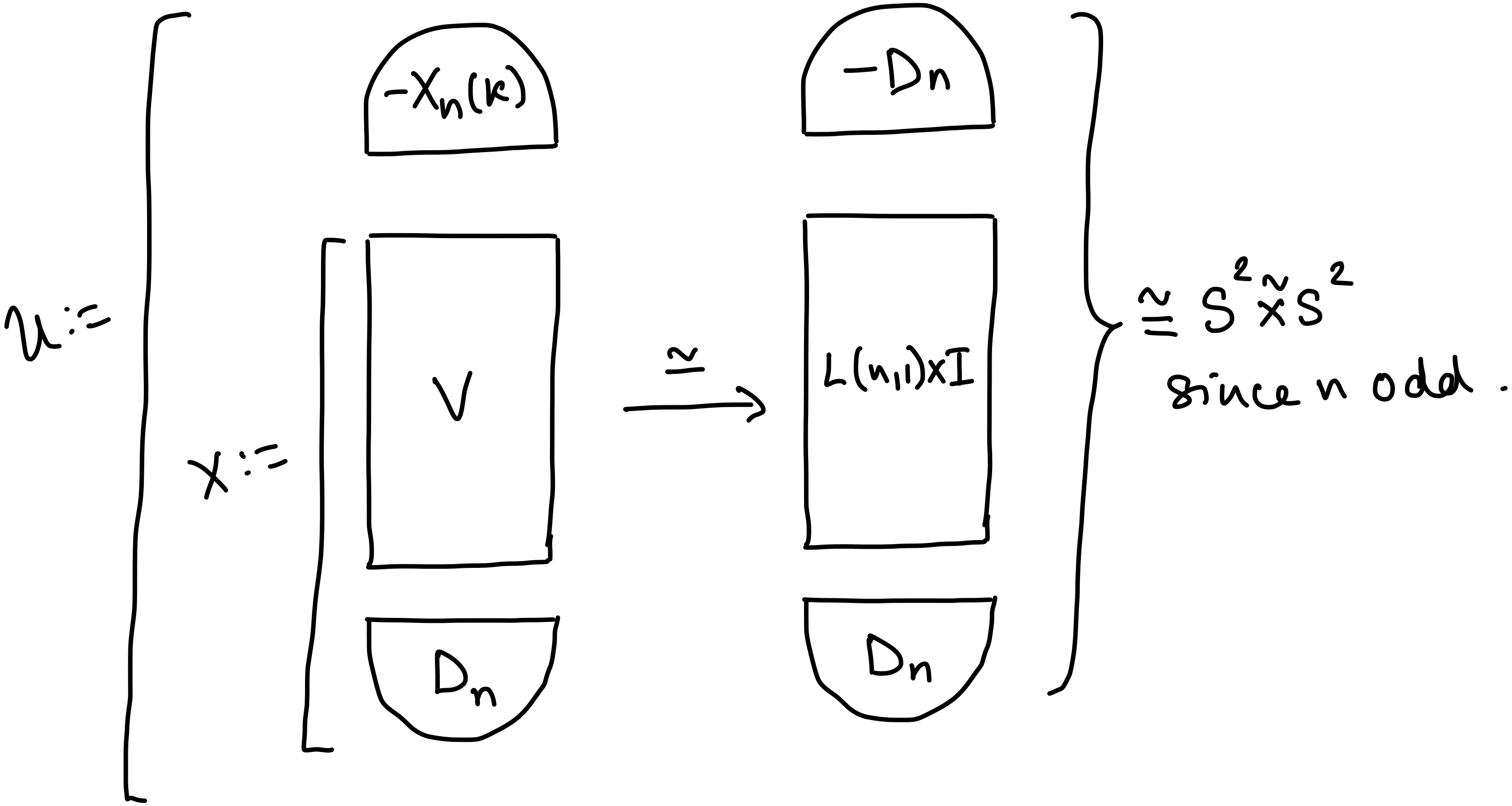
$$*CP^2 \not\cong CP^2$$

$\tau(\text{gen of } \pi_2(*CP^2)) = 1$

$\tau(\text{gen of } \pi_2(CP^2)) = 0$

Recall our Goal: n odd, $X := V \cup D_n$, $ks(X) = Arf(K)$.





Use τ to show

$$\mathrm{Arf}(K) = 0 \iff \mathcal{U} \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \iff Rs(\mathcal{U}) = 0 \iff Rs(X) = 0$$

$$\mathrm{Arf}(K) = 1 \iff \mathcal{U} \cong *\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \iff Rs(\mathcal{U}) = 1 \iff Rs(X) = 1$$

Questions?