KU LEUVEN

Random quantum graphs are asymmetric

8th ECM, Portorož

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1 Outline

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Point of view for today: quantum graphs are interesting mathematical objects in their own right.

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Are these definitions equivalent?

1 The equivalences

Operator systems vs projections

Let $p : M_n \to V$ be the orthogonal projection wrt the trace. As $\mathrm{B}(\mathrm{HS}_n) \simeq M_n \otimes M_n^{\mathrm{op}}$, we get a corresponding $P \in M_n \otimes M_n^{\mathrm{op}}$.

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If $A : M_n \to M_n$ is cp and such that $m(A \otimes A)m^* = A$ then $P_A := (A \otimes \mathrm{Id})m^*(\mathbb{1})$, its Choi matrix, is a projection in $M_n \otimes M_n^{\mathrm{op}}$.

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$(A \otimes \mathrm{Id})m^*(\mathbb{1}) = \frac{1}{n} \sum_{i,j=1}^{n} A(e_{ij}) \otimes e_{ji}$

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If the spectrum of D is simple then the automorphism group is automatically abelian.

2 Outline

2 Random models

The easiest thing to do is the following. We fix $0 \leqslant d \leqslant n^2 - 1$. Then we take d independent random Hermitian matrices X_1,...,X_d and consider $V_d := \mathrm{span}\{\mathbb{1}, X_1, \ldots, X_d\}$.

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$G(n, M)$

The model above corresponds to the Erdős-Rényi random graph $G(n, M)$ with a fixed number of edges. We can also build a version of the $G(n, p)$, where we fix the number of vertices and the probability that a given pair of vertices is connected by an edge.

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Theorem (Chirvasitu-W.)

If $2 \leqslant d \leqslant n^2 - 3$ *then the automorphism group is almost surely trivial.*

What about quantum symmetries?

Thank you for your attention!