Generalization of proofs and codification of graph families

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8ECM

Contents

Fermat numbers

- n-th Fermat number: $F(n) := 2^{2^n} + 1$.

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- Only $F(0)$, ..., $F(11)$ are fully factored.
- $F(20)$ and $F(24)$ are known to be composite, but no prime factor is known.

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$$A \mid D + B \qquad A \mid D \cdot C + B \cdot C$$

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- Original calculation: $\left\{ \begin{array}{l} 641 \mid 5^4 + 2^4 \\ 641 \mid 5^4 \cdot 2^{28} - 1 \end{array} \right. \Rightarrow 641 \mid 2^{32} + 1.$

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- Particularization: $\boxed{k \cdot 2^s + 1 \mid k^{2 \cdot r} + 2^{2^n - 2 \cdot r \cdot s} \Rightarrow k \cdot 2^s + 1 \mid F(n)}$
  ($A = k \cdot 2^s + 1$, $B = 2^{2^n - 2 \cdot r \cdot s}$, $C = 2^{2 \cdot r \cdot s}$, $D = k^{2 \cdot r}$, $E = 1$).
  Note that, by varying r, we can make $k^{2 \cdot r} + 2^{2^n - 2 \cdot r \cdot s}$ way smaller than $F(n)$.

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  Note that, by varying r, we can make $k^{2 \cdot r} + 2^{2^n - 2 \cdot r \cdot s}$ way smaller than $F(n)$.
- Remark: this method can be applied in any area of mathematics.

Other results

- Result 1: $\boxed{\begin{cases} d \mid k \cdot 2^s \\ (k \cdot 2^s + 1) \cdot d^{2 \cdot t} - (k \cdot 2^s)^{2 \cdot t} + 1 = F(n) \end{cases} \Rightarrow \; k \cdot 2^s + 1 \mid F(n)}$.

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- Result 2: $\boxed{(k \cdot 2^s + 1 - k^{2 \cdot q}) \cdot 2^{2 \cdot q \cdot s} + 1 = F(n) \ \Rightarrow\ k \cdot 2^s + 1 \mid F(n)}$.
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- Result 3: $p \mid (F(n-1)-1-p\cdot c)^2+1\ \Rightarrow\ p \mid F(n)$.
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- Result 3: $\boxed{p \mid (F(n-1) - 1 - p \cdot c)^2 + 1 \ \Rightarrow\ p \mid F(n)}$.
  Note that, by varying c, we can make $(F(n-1) - 1 - p \cdot c)^2 + 1$ way smaller than $F(n)$.
- Result 4: $\boxed{p \nmid \dfrac{(F(n+2) - 1)^k - 1}{F(n)} \ \Leftrightarrow\ p \mid F(n)}$,
  where p equals $k \cdot 2^{n+2} + 1$ and is prime.
  The converse implication was proved by Wang.

Contents

Vertex-by-vertex increasing graph sequences

- A sequence G of simple graphs is vertex-by-vertex increasing if and only if $V(G_n) = \{1, ..., n+1\}$ and $E(G_n) \subseteq E(G_{n+1})$.

The bijection

Input: vertex-by-vertex increasing graph sequence G such that G_∞ is Collatz's graph.

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G_1	$\mathrm{adj}(G_1)$	
1 —— 2	$\begin{bmatrix} * & * \\ 1 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$

The bijection

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G_2	$\mathrm{adj}(G_2)$	
1 —— 2 3	$\begin{bmatrix} * & * & * \\ 1 & * & * \\ 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$

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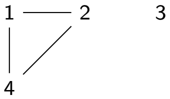
G_3	$\mathrm{adj}(G_3)$	
	$\begin{bmatrix} * & * & * & * \\ 1 & * & * & * \\ 0 & 0 & * & * \\ 1 & 1 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$

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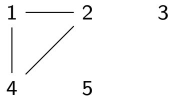
<table>
<tr><th>G_4</th><th>$\mathrm{adj}(G_4)$</th><th></th></tr>
<tr><td>
</td><td>$$\begin{bmatrix} * & * & * & * & * \\ 1 & * & * & * & * \\ 0 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$</td><td>$(\Phi(G))_0 = 1$
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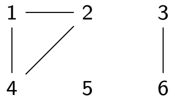	$\begin{bmatrix} * & * & * & * & * & * \\ 1 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 1 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$ $(\Phi(G))_3 = 0$ $(\Phi(G))_4 = 4$

The bijection

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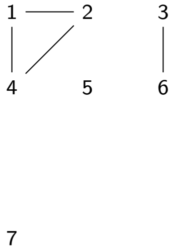
G_6	$\mathrm{adj}(G_6)$	
	$\begin{bmatrix} * & * & * & * & * & * & * \\ 1 & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 1 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$ $(\Phi(G))_3 = 0$ $(\Phi(G))_4 = 4$ $(\Phi(G))_5 = 0$

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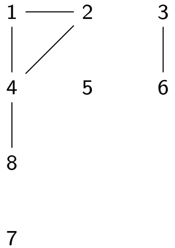
G_7	$\mathrm{adj}(G_7)$	
	$\begin{bmatrix} * & * & * & * & * & * & * & * \\ 1 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 1 & 1 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$ $(\Phi(G))_3 = 0$ $(\Phi(G))_4 = 4$ $(\Phi(G))_5 = 0$ $(\Phi(G))_6 = 8$

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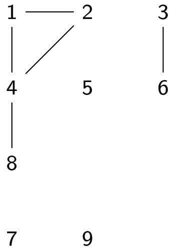	$\begin{bmatrix} * & * & * & * & * & * & * & * & * \\ 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 1 & 1 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$ $(\Phi(G))_3 = 0$ $(\Phi(G))_4 = 4$ $(\Phi(G))_5 = 0$ $(\Phi(G))_6 = 8$ $(\Phi(G))_7 = 0$

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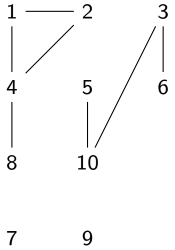	$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ 1 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * \\ 1 & 1 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & * \end{bmatrix}$	$(\Phi(G))_0 = 1$ $(\Phi(G))_1 = 0$ $(\Phi(G))_2 = 6$ $(\Phi(G))_3 = 0$ $(\Phi(G))_4 = 4$ $(\Phi(G))_5 = 0$ $(\Phi(G))_6 = 8$ $(\Phi(G))_7 = 0$ $(\Phi(G))_8 = 80$

Application to Collatz Problem

- Output: $(\Phi(G))_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \sqrt{2^n} + \sqrt[3]{4^{n+1}}, & \text{if } n \in \{6 \cdot r + 2\}_{r \in \mathbb{N}}; \\ \sqrt{2^n}, & \text{otherwise,} \end{cases}$
  (first terms: 1, 0, 6, 0, 4, 0, 8, 0, 80...).

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- Collatz Problem: is G_∞ connected?
- Pattern observed: if $n \geq 3$, then the number of connected components of G_n is apparently $\left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+3}{6} \right\rfloor + 1$.

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- Pattern observed: if $n \geq 3$, then the number of connected components of G_n is apparently $\left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+3}{6} \right\rfloor + 1$.
- Remark: because Φ is a bijection, we can input integer sequences instead (provided that they are upper-bounded by 2^{n+1}).