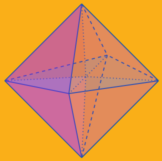
Enumeration and universality classes of colored triangulations

(for quantum gravity)



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LIPN – Sorbonne Paris Nord
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June 23-24, 2021 **8ECM**

Applied Combinatorial and Geometric Topology

(MS - ID 34)

First half

- Trees and combinatorial maps
- Universality classes
- Enumeration and scaling limit
- Model for questions in higher dimensions

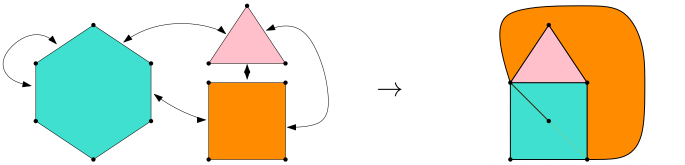
Second half

- Some 3D models
- Enumeration for colored triangulations which maximize the number of $(d-2)$-simplices at fixed number of d-simplices
  [VB + others mentioned in title slide]

Review material except last part

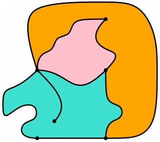
Combinatorial maps

Take polygons and glue into closed surface

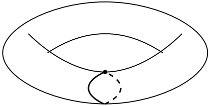
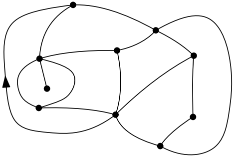


Properly embedded graphs

- No crossing & up to deformation
- Graph complement is disjoint union of disks
- Each disk: a **face**



Maps

- ▷ Not a map  Planar map 
- ▷ Vertices, edges and faces
- ▷ Other def: factorization of permutations $\rightarrow$ Hurwitz numbers
- ▷ **Euler's formula**: Genus h is determined combinatorially

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = 2 - 2h, \qquad h \geq 0$$

- ▷ For triangulations $V \leq 2 + F/2$ Linear bound on number of vertices

Equivalence in 2D

$$\text{Sphere} \Leftrightarrow \text{Genus } 0 \Leftrightarrow \text{Maximize } V \text{ at fixed } F$$

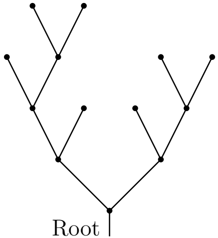
Not true in higher dimensions!

Families of maps and trees

- General maps, triangulations, quadrangulations, bipartite maps, hypermaps
- **Enumeration** [Tutte 60s] $Q_n = \frac{2\cdot 3^n}{(n+2)(n+1)}\binom{2n}{n} \sim_{n\to\infty} \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}$
- **Geometry**: graph distance, dual graph distance, etc. How do distances scale with the size? (very active topic last 20 years)

Trees

- Graphs without cycle
- Enumeration w.r.t. number (internal) nodes
- Also geometric objects: graph distance
- Also multiple families: plane trees, binary, etc.



- Do trees or maps from different families behave *similarly*? Not exactly but at large scales?

Universality

- Common asymptotic and geometric features?
- Macroscopic quantities **independent** of microscopic details of model
- Develop universal features at large scales
- We do not see the individual polygons anymore
- Collective behavior due to gluing
- For "reasonable" family $\mathcal{T}$ of trees

$$T_n \sim \underbrace{K_{\mathcal{T}}}_{\text{Constant}} \quad \underbrace{\rho_{\mathcal{T}}^{-n}}_{\text{Exp growth}} \quad n^{-3/2} \qquad \begin{cases} K_{\mathcal{T}} \& \rho_{\mathcal{T}} \text{ non-universal} \\ \textbf{exponent } -3/2 \text{ universal} \end{cases}$$

- For reasonable family of maps $\mathcal{M}$, $\qquad M_n \sim K_{\mathcal{M}} \; \rho_{\mathcal{M}}^{-n} \; n^{-5/2}$

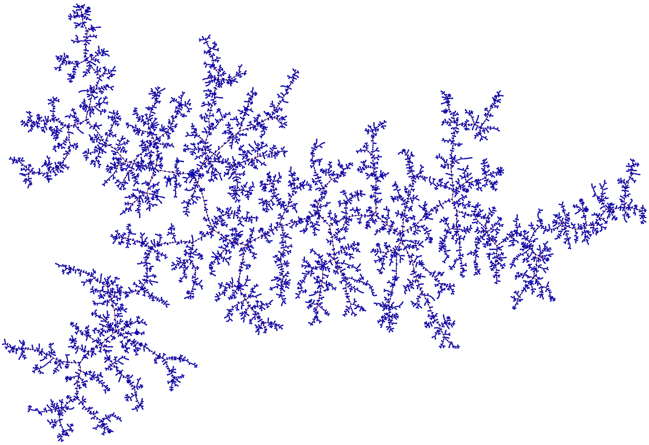
Universality in the continuum limit

- Size $n \to \infty$: asymptotics
- Edge length $a \to 0$, so that distances remain finite: scaling behavior

$$a \sim n^{-1/2} \quad \text{for trees,} \qquad a \sim n^{-1/4} \quad \text{for maps}$$

Scaling limit of random trees

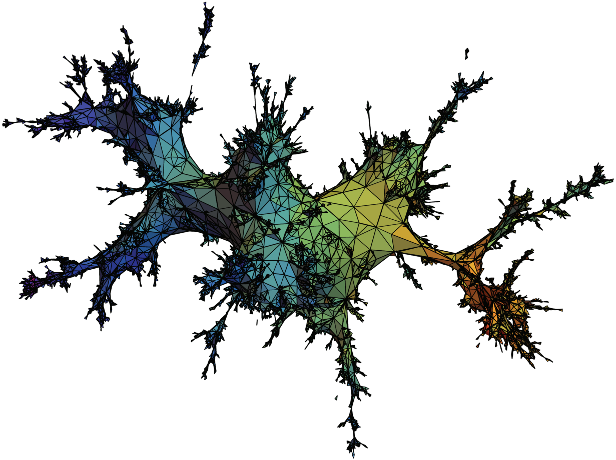
- Aldous' thm: convergence to the **continuous random tree**
- Distances scale like $\sqrt{n}$, Hausdorff dim $d_H = 2$
- Almost surely binary



Simulation by I. Kortchemski

Scaling limit of random planar maps

- Le Gall's & Miermont's thm: convergence to the **Brownian sphere**
- Distances scale like $n^{\frac{1}{4}}$, Hausdorff dim $d_H = 4$
- Almost surely homeomorphic to 2-sphere



Simulation by I. Kortchemski – 3D rendering on Bettinelli's page

Some classes of random geometry

- Watabiki's prediction $d_H = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$

Species	**Asymptotics**	d_H	**Scaling limit**
Trees	$n^{-3/2}$	2	Continuous random tree [Aldous]
Planar	$n^{-5/2}$	4	Brownian sphere [Le Gall & Miermont]
Planar + spanning tree	n^{-3}	$\frac{3+\sqrt{17}}{2}$?	??
Planar + Ising	$n^{-7/3}$	$\frac{7+\sqrt{97}}{4}$?	Huge open problem
Meanders	$n^{-\frac{29+\sqrt{145}}{12}}$?		Conjecture [Di Francesco-Golinelli-Guitter]
Feuilletages		$(2^k)_{k\geq 1}$	[Marckert-Lionni]

In higher dimensions

- Difficult to find a model both solvable and non-trivial!
- Topology and combinatorics of triangulations is more difficult
- Can we recover results from 3D gravity by summing triangulations?
- Universality required if this is viable approach to **quantum gravity**

Some difficulties

- No topological classification using a single integer
- Euler characteristic vanishes for all orientable 3-manifolds
- Matrix integral approach generalize to tensor integrals, but no eigenvalues

- Setup to investigate universality
- Generalize triangulations, quadrangulations, etc. to higher dim. families
- Need to generalize polygons to building blocks
- Still maintaining combin. control enough to enumerate

3-spheres

"Gromov" question

- Is the number of 3d triangulations of the sphere **exponentially bounded** w.r.t. number of tetrahedra?
- Rivasseau's bound $n!^{1/3}$ using colored triangulations
- Chapuy-Perarnau's refinement $n!^{1/6}$ Exact for 3-manifolds!
- Numerical simulations and sub-families...

Numerical simulations – Euclidean Dynamical Triangulations

- Monte-Carlo simulations found two phases
- A phase which maximizes the number of edges corresponding to trees
- A phase which minimizes the number of edges with few vertices (crumbled)

Sub-families of triangulations

Stack-spheres

- Built by $1 \to 4$ moves on tetrahedra at random
- Scaling limit is Aldous' CRT [Albenque-Marckert, Gurau-Ryan]

Locally constructible

- Take a tree of tetrahedra and only glue adjacent faces
- **Locally constructible** triangulations [Durhuus-Jonsson]
- LCT are exponentially bounded
- Benedetti & Ziegler proved that not all spheres are LC

$$\left\{ \begin{matrix} \text{vertex} \\ \text{decomposable} \end{matrix} \right\} \subsetneq \{\text{shellable}\} \subseteq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{3-spheres}\}$$

- Monte-Carlo in progress [private comm. Budd-Lionni]

Combinatorial classification

- Euler's genus formula for $2p$-angulations

$$V(T) - (p-1)F(T) = 2 - 2g(T) \leq 2$$

- Which set of triangulations to generalize? **Colored triangulations**

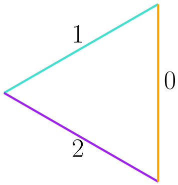
Gurau's theorem on colored triangulations

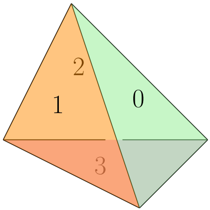
- Bound on number of $(d-2)$-simplices

$$\Delta_{d-2}(T) - \frac{d(d-1)}{4}\Delta_d(T) = d - \omega(T) \leq d$$

- $\omega(T) = 2g(T)$ for 2D colored triangulations
- Not a topological invariant in $d \geq 3$
- Genuine **combinatorial** extensions of genus!
- Gurau–Schaeffer classification w.r.t. $\omega(T)$
- Investigate universality classes?

Colored simplex



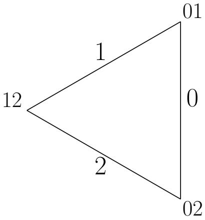


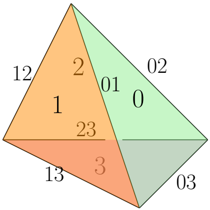
Induced colorings

- Faces colored $0, 1, \ldots, d$

Colors identify all sub-simplices

Colored simplex



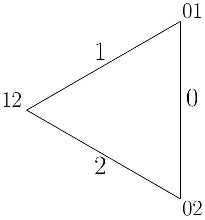


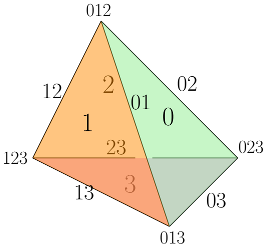
Induced colorings

- Faces colored $0, 1, \dots, d$
- $(d-2)$-simplices labeled by pairs of colors $\{a, b\}$

Colors identify all sub-simplices

Colored simplex



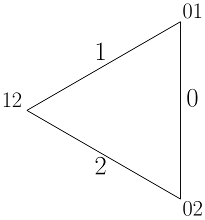


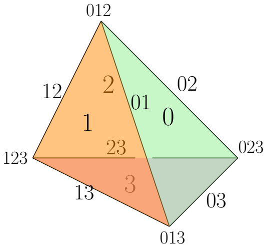
Induced colorings

- Faces colored $0, 1, \ldots, d$
- $(d-2)$-simplices labeled by pairs of colors $\{a, b\}$
- $(d-3)$-simplices labeled by triples of colors $\{a, b, c\}$

Colors identify all sub-simplices

Colored simplex





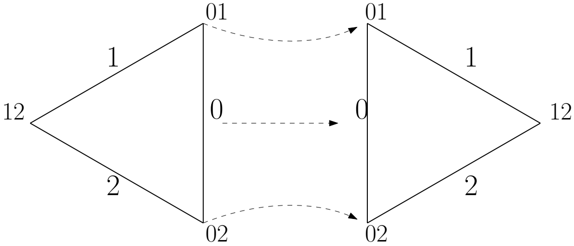
Induced colorings

- Faces colored $0, 1, \ldots, d$
- $(d-2)$-simplices labeled by pairs of colors $\{a, b\}$
- $(d-3)$-simplices labeled by triples of colors $\{a, b, c\}$
- $(d-k)$-simplices labeled by k-uple of colors

Colors identify all sub-simplices

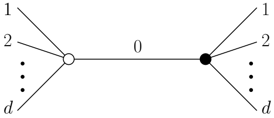
Attaching map

Unique gluing which respects all subcolorings



Gluing determined by face color

- Graphical representation



Attaching map

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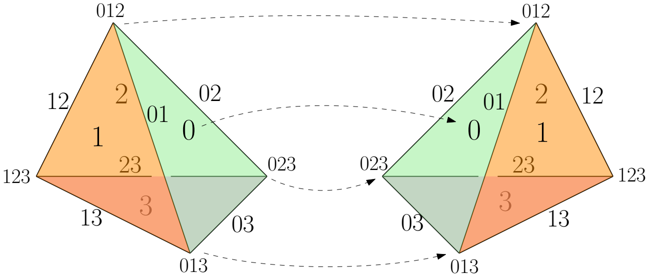

Gluing determined by face color

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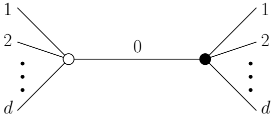
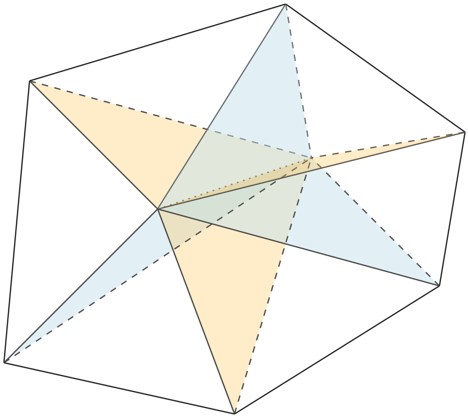


$(d-2)$-simplices and bicolored cyles

$d = 3$



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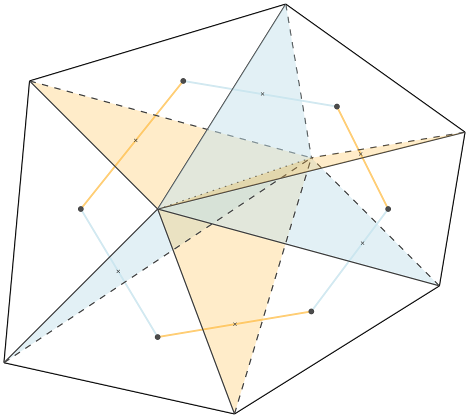
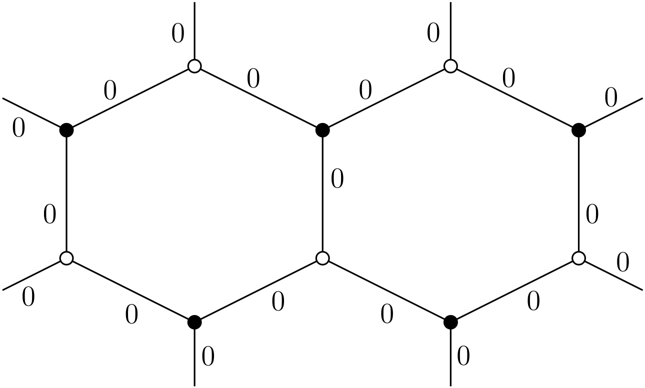
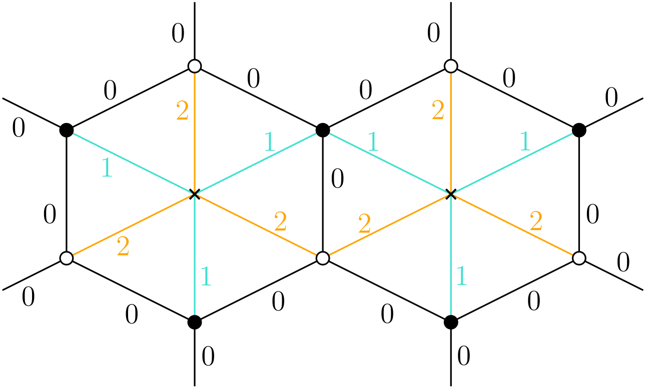


Revisiting the 2D case

Bipartite $2p$-angulations



Bipartite $2p$-angulations



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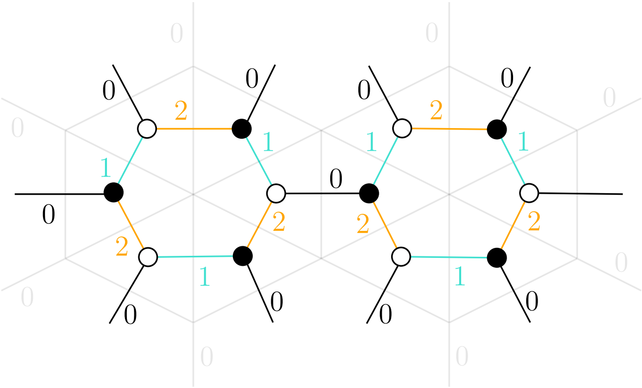
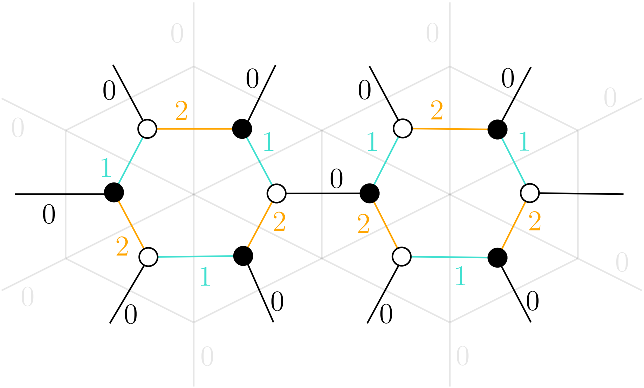


Revisiting the 2D case

Bipartite $2p$-angulations

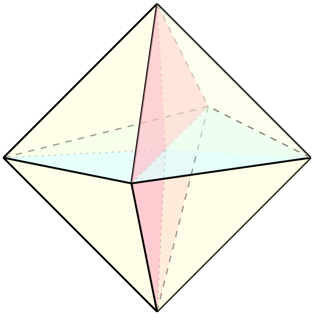


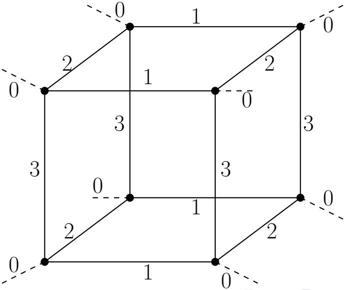
- ▷ $2p$-gon is dual to a cycle with colors $\{1, 2\}$
- ▷ Color 0 used to glue $2p$-gons together

Generalizing polygons

Colored building blocks and dual bubbles

- Boundary made of $(d-1)$-simplices of color 0
- Cone over bdry triangulation $\to$ 1 internal vertex, of colors $1, \ldots, d$
- Dual graph called **bubble**: connected, all colors except 0
- Edges of color 0 glue bubbles together
- Investigate universality!





Some theorems

- ▷ All PL-manifolds admit a colored triangulation
- ▷ See following talks by P. Cristofori and M. R. Casali
- ▷ Represent a manifold iff c.c. obtained by removing a color are all spheres
- ▷ Orientability iff colored graph is bipartite
- ▷ Topological studies ([L. Grasselli,P. Cristofori and M. R. Casali]) but little enumeration until 2010!

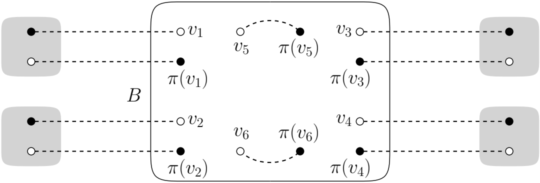
In combinatorics and math-ph

- ▷ Tensor integral to generate colored triangulation via Feynman rules (GFT context [Gurau])
- ▷ Large N limit and combinatorial extension of the genus to higher d
- ▷ More large N limits in tensor models and connection to SYK model [Benedetti-Carrozza-Ferrari-Gurau-Harribey-Klebanov, etc.]
- ▷ Some universality classes [VB-Lionni-Thürigen]
- ▷ Beyond large N, e.g. higher genus & topological recursion
- ▷ GFT renormalization at all orders in perturbation theory

Enumeration of 3D colored triangulations

Thm [VB]

- Take any set of colored building blocks homeomorphic to 3-balls
- Which gluings **maximize the number of edges**?
- Topologically: 3-spheres Combinatorially: bijection with trees



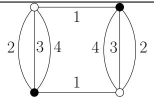
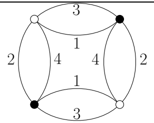
- What about the bound? Example: octahedra [VB-Lionni]

$\mathrm{Edges}(T) \leq 3 + 11\mathrm{Oct}(T)$ vs Gurau's $\mathrm{Edges}(T) \leq 3 + 12\mathrm{Oct}(T)$

- Bound is not universal
- Triangulations which saturate the bond exhibit **universality**

Surprise

- In even dim. universality class depends on building blocks
- $d = 4$

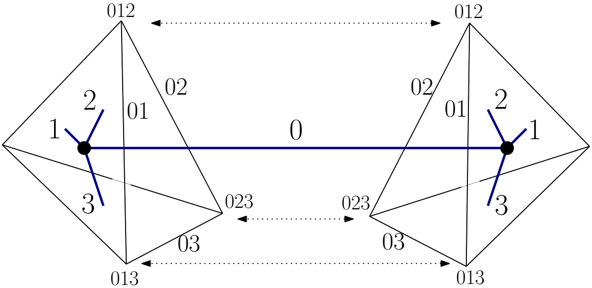
<table>
<tr><th>Building block</th><th>Maximize # triangles</th><th></th></tr>
<tr><td>
</td><td>Trees</td><td></td></tr>
<tr><td>
</td><td>Planar maps</td><td>Reproduce all 2D behaviors</td></tr>
<tr><td>Mix</td><td>Trees of baby universes</td><td></td></tr>
</table>

Summary

- Investigate universality classes of discretized PL-manifolds
- Combinatorial maps = 2D quantum gravity
- Colored triangulations are nice objects to investigate universality classes in higher dimensions
- Genuine combinatorial generalization of Euler's relation
- Maximize number of $(D-2)$-simplices
- No new universality classes (yet)
- Need different guiding principles, more focus on topology?
- Other families offering nice meeting ground for topology and combinatorics?
- Geometry of colored triangulations?

Dual colored 1-skeleton

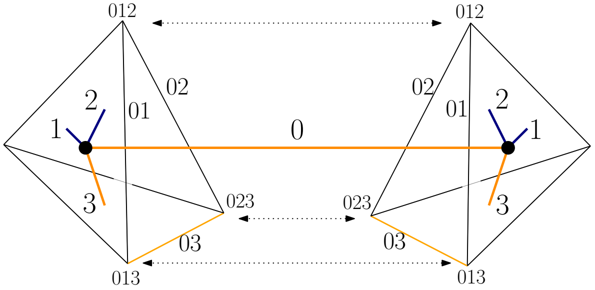
Colored Duality



Triangulation	Graph
d-simplex	Vertex
$(d-1)$-simplex of color c	Edge of color c

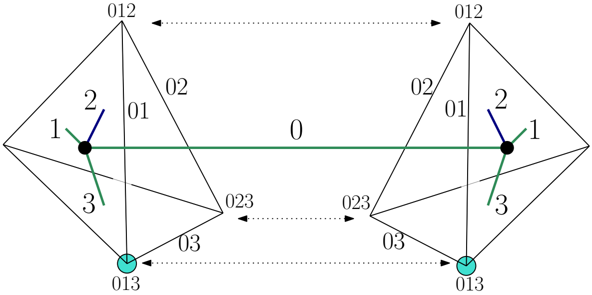
Dual colored 1-skeleton

Colored Duality



Triangulation	Graph
d-simplex	Vertex
$(d-1)$-simplex of color c	Edge of color c
$(d-2)$-simplex with 2 colors	Bicol cycle

Colored Duality

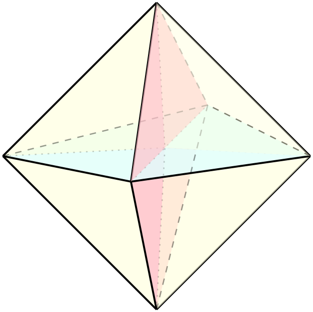


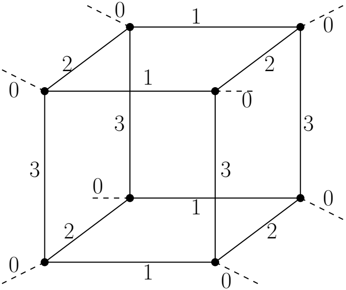
Triangulation	Graph
d-simplex	Vertex
$(d-1)$-simplex of color c	Edge of color c
$(d-2)$-simplex with 2 colors	Bicol cycle
$(d-3)$-simplex with 3 colors	Tricol c.c.

$(d-3)$-simplices and tricolored components

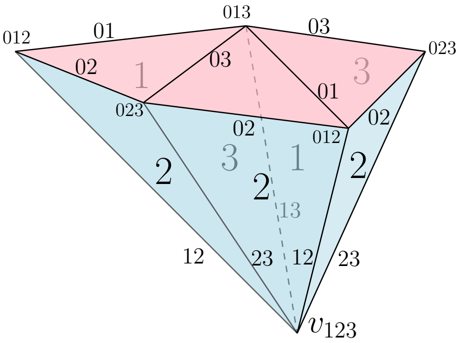
$d = 3$

- Octahedron made of 8 colored tetrahedra
- All boundary faces have color 0
- All internal triangles have colors 1, 2, 3
- Single internal vertex of color $\{1, 2, 3\}$
- Dual: cube with colored edges 1, 2, 3





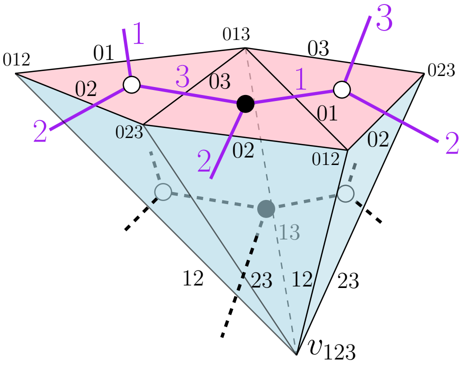
Colored Building blocks



Cones

- Boundary $\Leftrightarrow$ Faces of color 0
- **Topological cones** over boundary
- Single interior vertex $v_{1 \ldots d}$

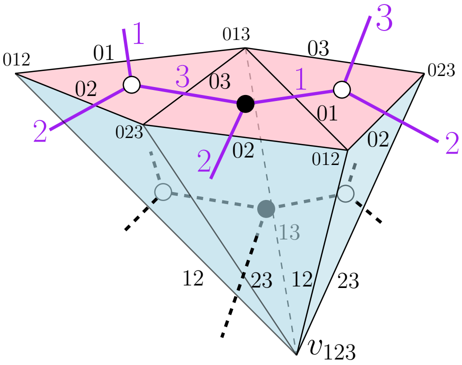
Colored Building blocks



Duals

- Dual to boundary = Dual to CBB with 0 removed
- Graph with colors $1, \ldots, d$ called **bubble**
- Same as colored graphs but without 0

Colored Building blocks

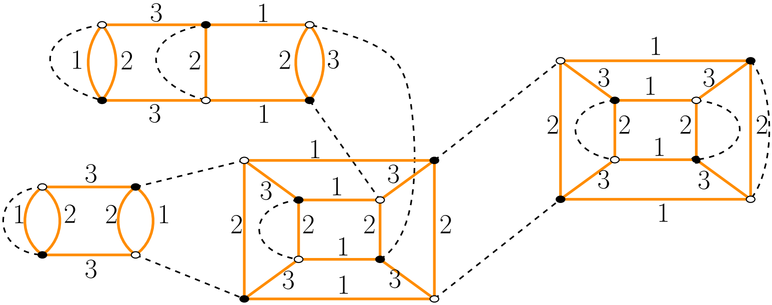


Proposition in 3D

- Dual to boundary is a map: Canonical embedding of bubble
- Colored building block **homeomorphic to 3-ball** $\Leftrightarrow$ **Planar** bubble
- We will be able to use properties of planar maps!

Example of graphs

In 3D graphs are 4-regular with colored edges

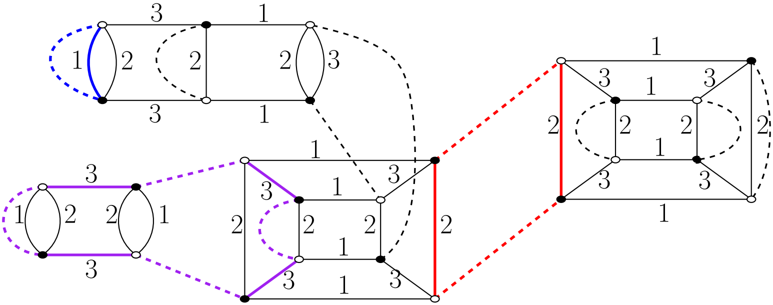


- $\triangleright$ 4 c.c. with colors $\{1, 2, 3\} \to$ 4 vertices of colors $\{1, 2, 3\}$ in triangulation
- $\triangleright$ $C_{0a}(G)$ number of **bicolored cycles** with colors $\{0, a\}$

$$C_0(G) = \sum_{a=1}^{3} C_{0a}(G) \qquad \text{dual to } (d-2)\text{-simplices}$$

Example of graphs

In 3D graphs are 4-regular with colored edges



- 4 c.c. with colors $\{1, 2, 3\} \rightarrow$ 4 vertices of colors $\{1, 2, 3\}$ in triangulation
- $C_{0a}(G)$ number of **bicolored cycles** with colors $\{0, a\}$

$$C_0(G) = \sum_{a=1}^{3} C_{0a}(G) \qquad \text{dual to } (d-2)\text{-simplices}$$