Connected (n_k) configurations exist for almost all n

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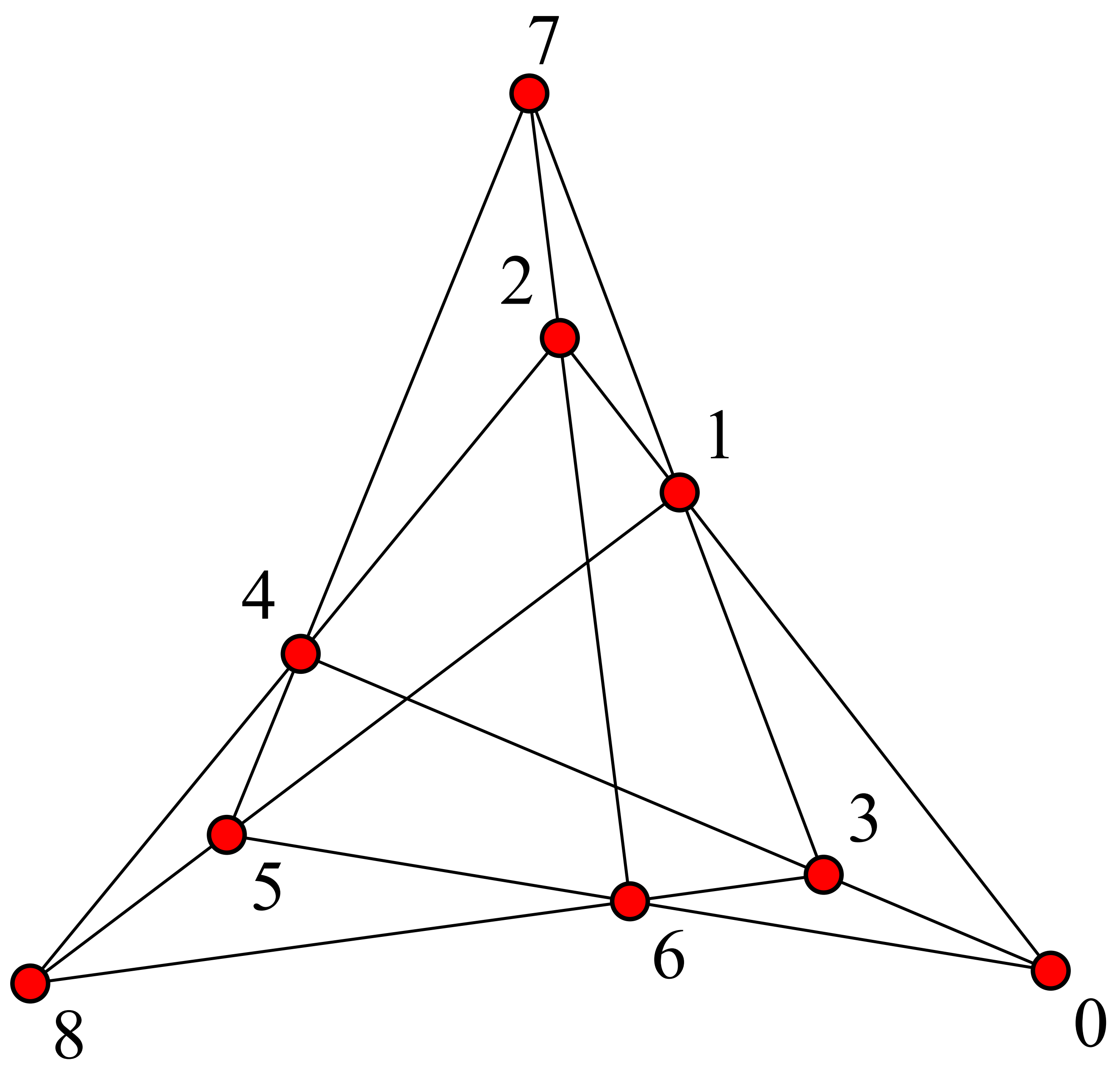
(geometric) *k*-configuration, (n_k) configuration



(9_3) configuration
3-configuration

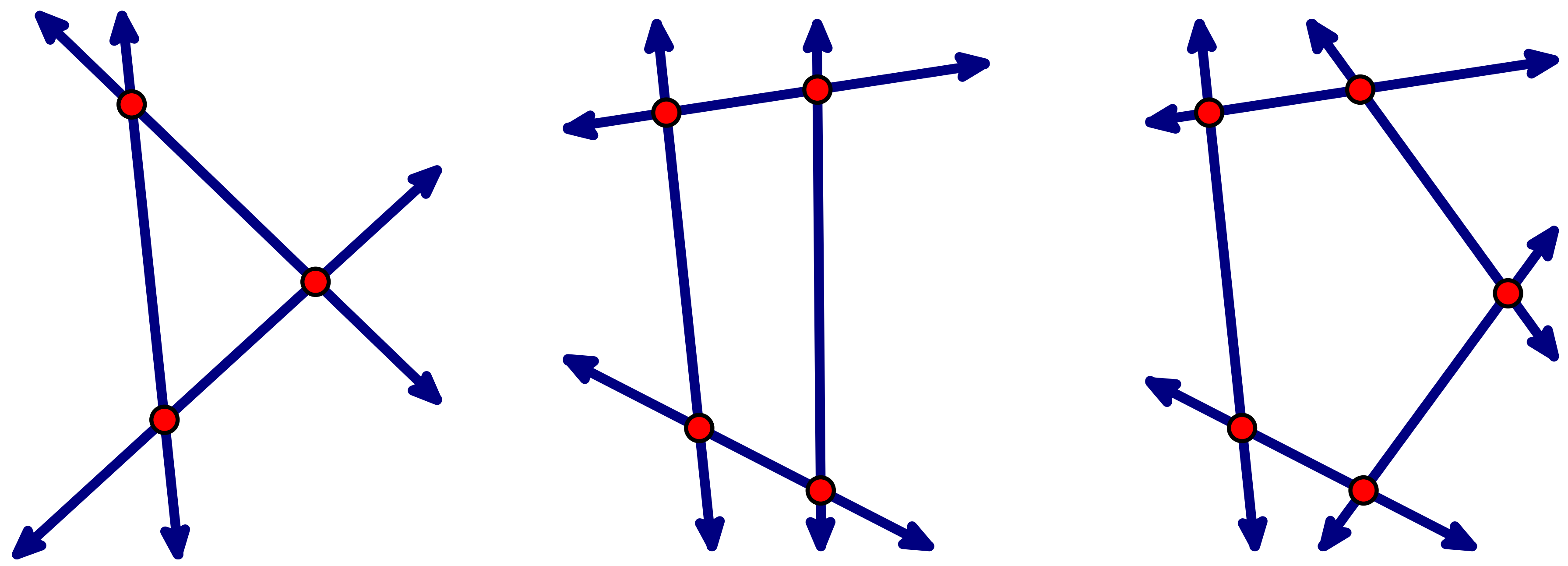
n points
n straight lines
k lines/point
k points/line

A	B	C	D	E	F	G	H	I
0	0	0	1	1	2	2	3	4
1	3	5	3	5	4	6	6	5
2	4	6	7	8	8	7	8	7

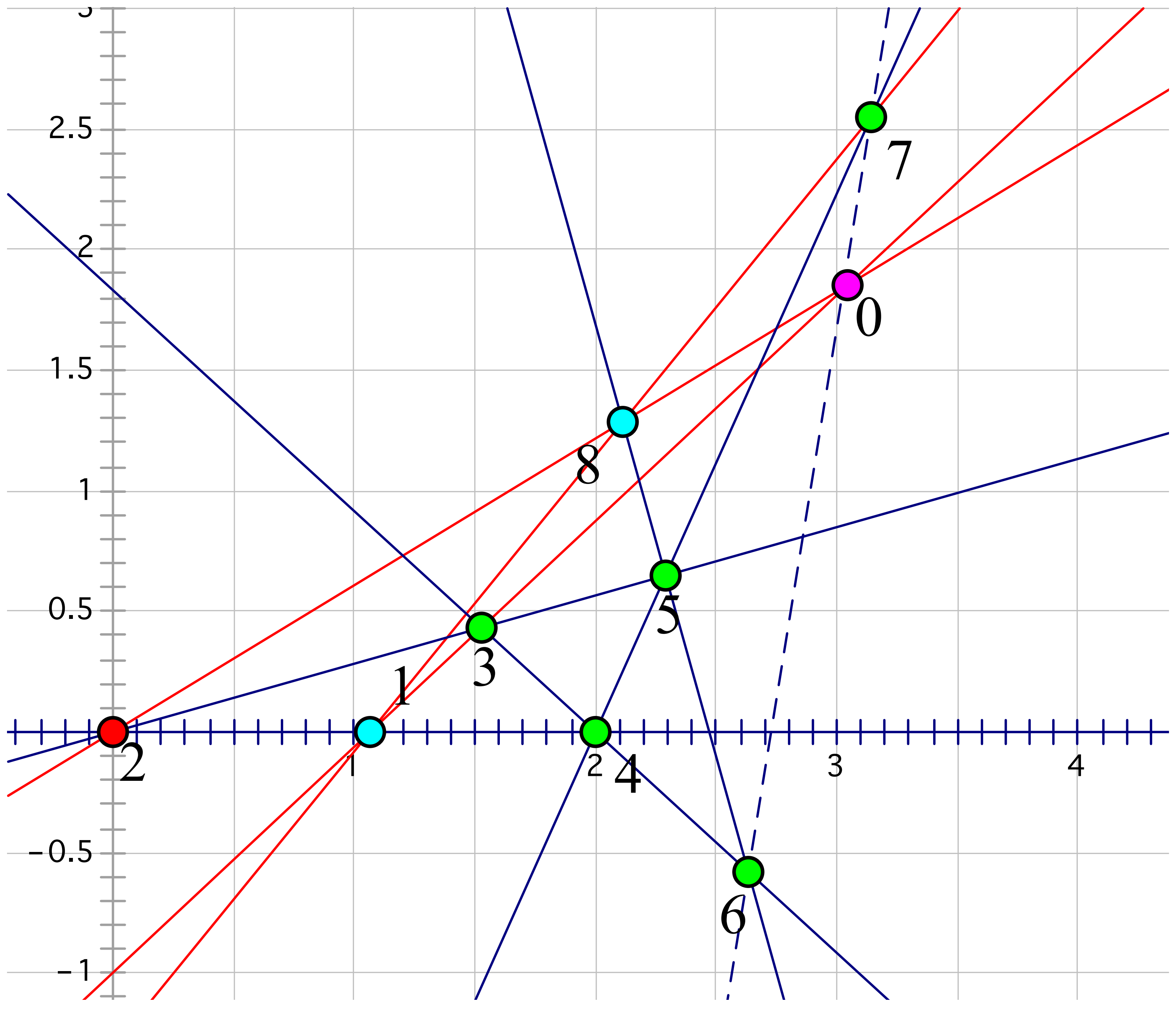
A Fundamental Question: For which n does there exist a geometric (n_k) configuration?

Does there exist some N_k so that for all $n \geq N_k$, there exists a geometric (n_k) configuration?

$n = 2$: $N_2 = 3$



$n = 3$: $N_3 = 9$



Cyclic configuration [0,1,3] for all $n \geq 9$

n=4: N_4 = 20 or 24

n	≤17	18	19	20	21	22	23	24	25	26	27…
#	NONE	2	NONE	≥1	≥1	≥1	?	≥1	≥1	≥1	≥1

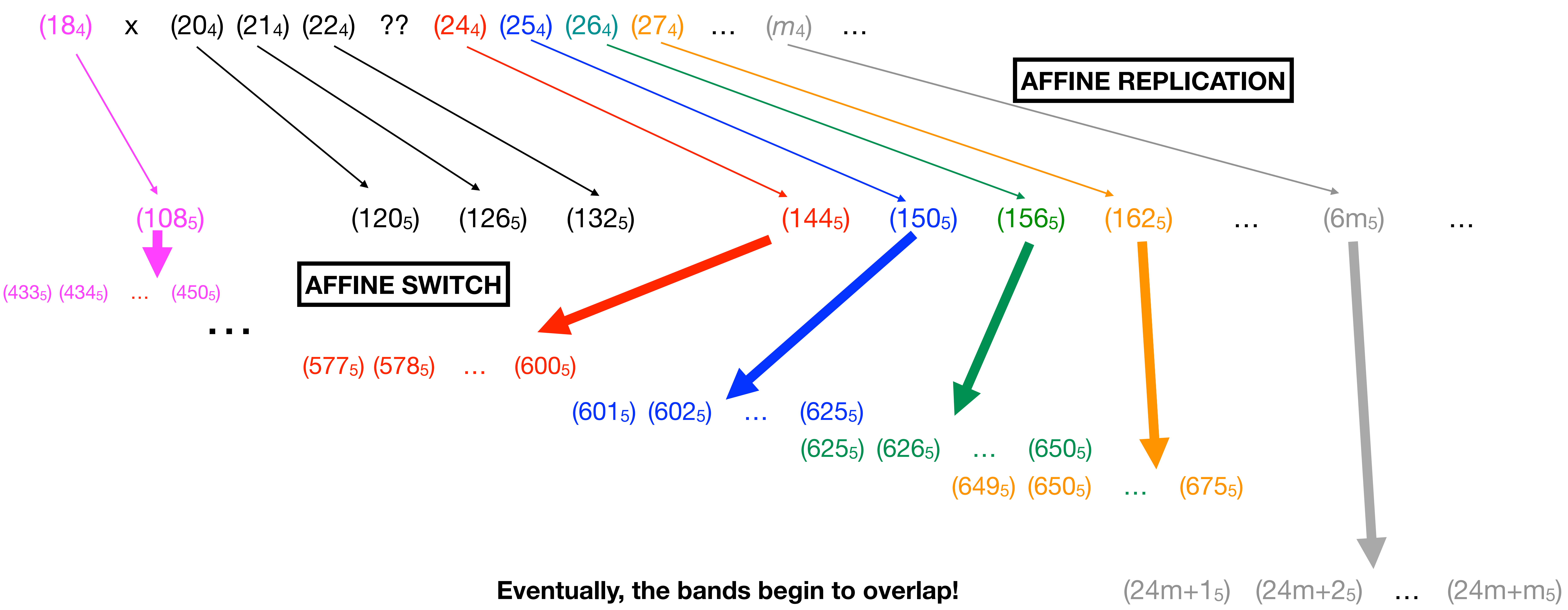
(Grünbaum 2000, 2002, 2006; 2009),
Bokowski & Shewe 2013, Bokowski & Pilaud 2015, 2016)
(Cuntz 2018)

$N_4 \leq 210$: combine two constructions
$24 \leq N_4 \leq 209$: ad hoc constructions

What can we say about N_5? N_k for larger k?

Systematic constructions:
“Grünbaum Incidence Calculus”

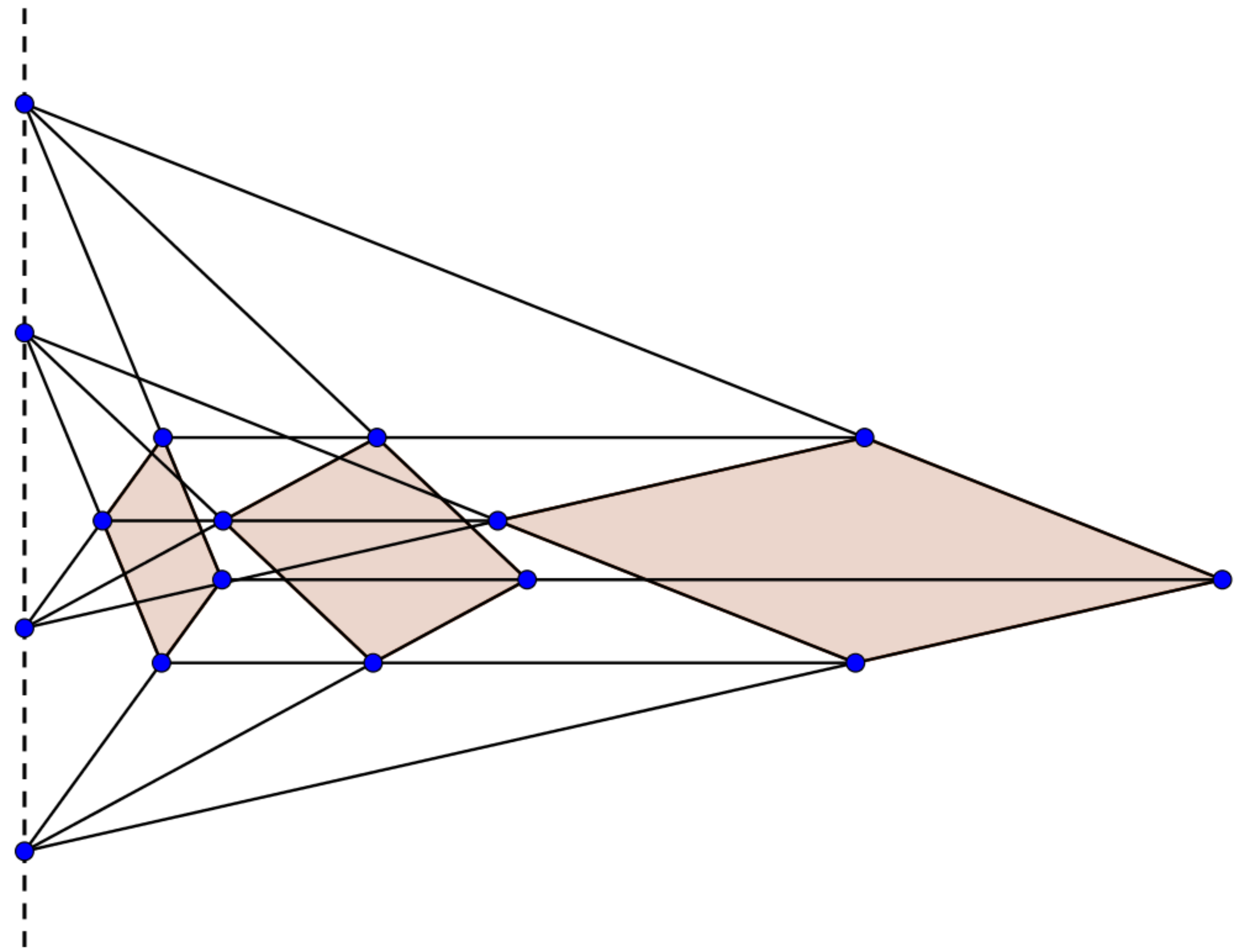
General idea



Affine Replication

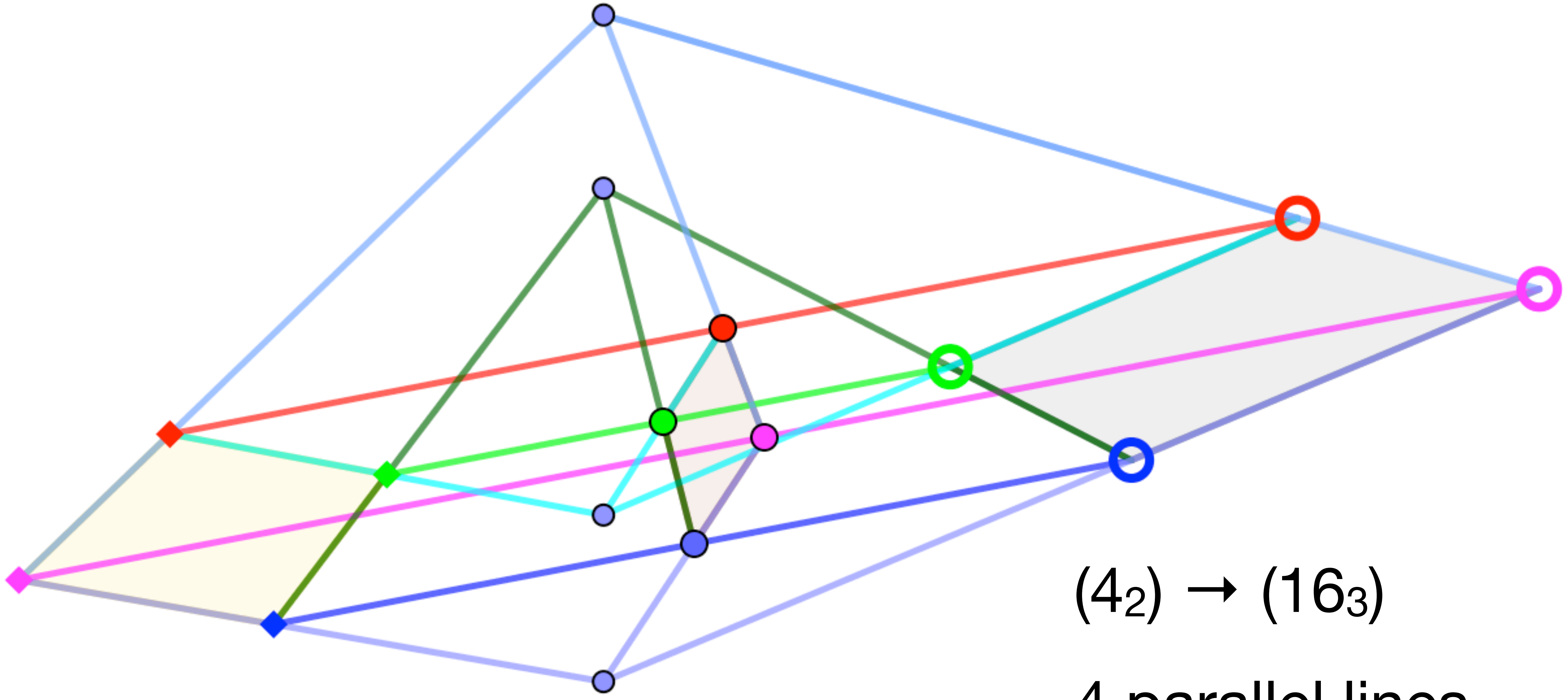
$(m_{k-1}) \rightarrow ((k+1)m_k)$ with m parallel lines

Construct k-1 copies of a configuration C so that corresponding image points are collinear, and corresponding image lines are concurrent!



Affine Replication

$(m_{k-1}) \rightarrow ((k+1)m_k)$ with m parallel lines



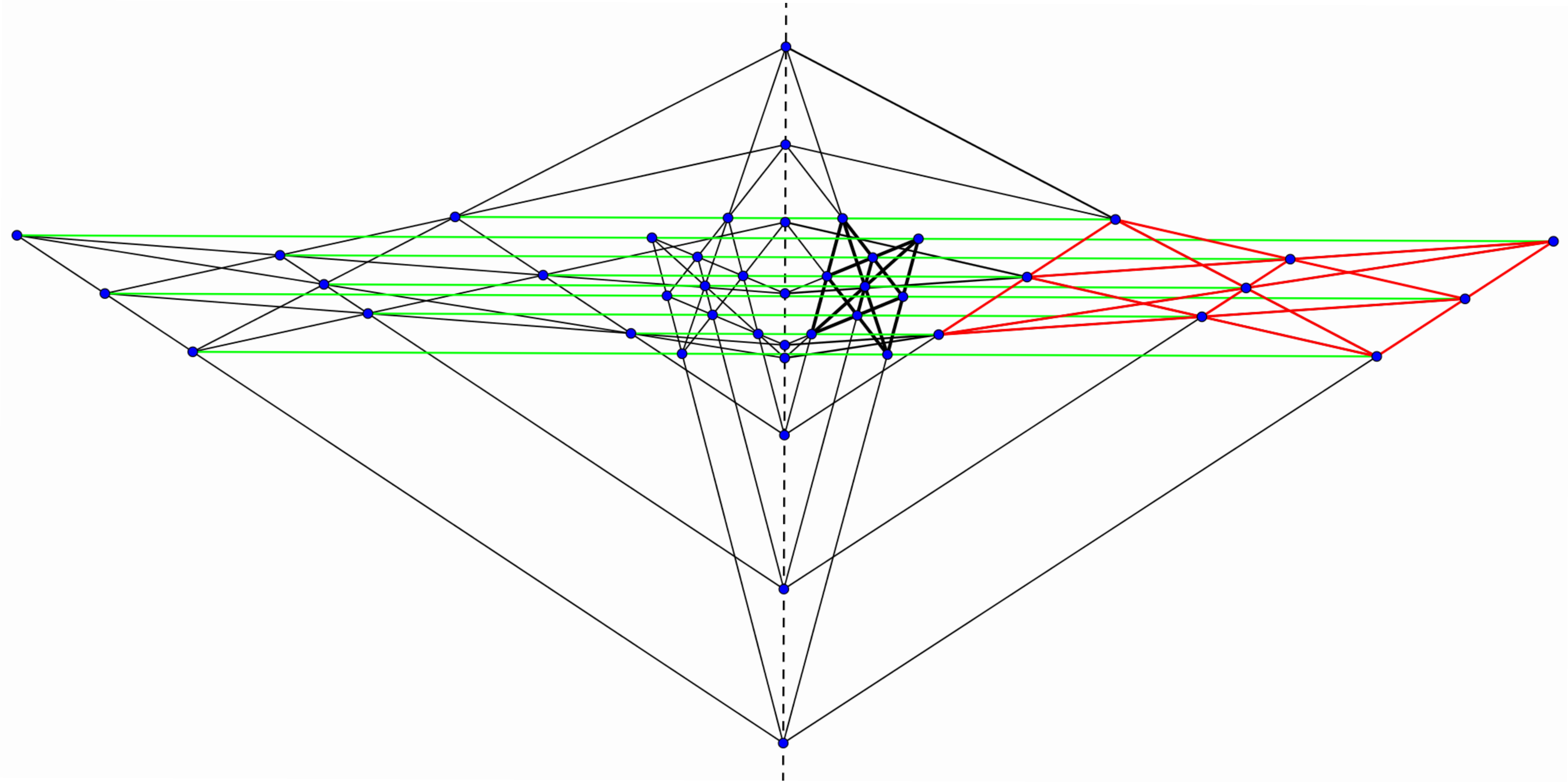
$(4_2) \rightarrow (16_3)$

4 parallel lines

Main Tool: Use k-1 axial affinities with the same axis and parallel vectors

Affine Replication

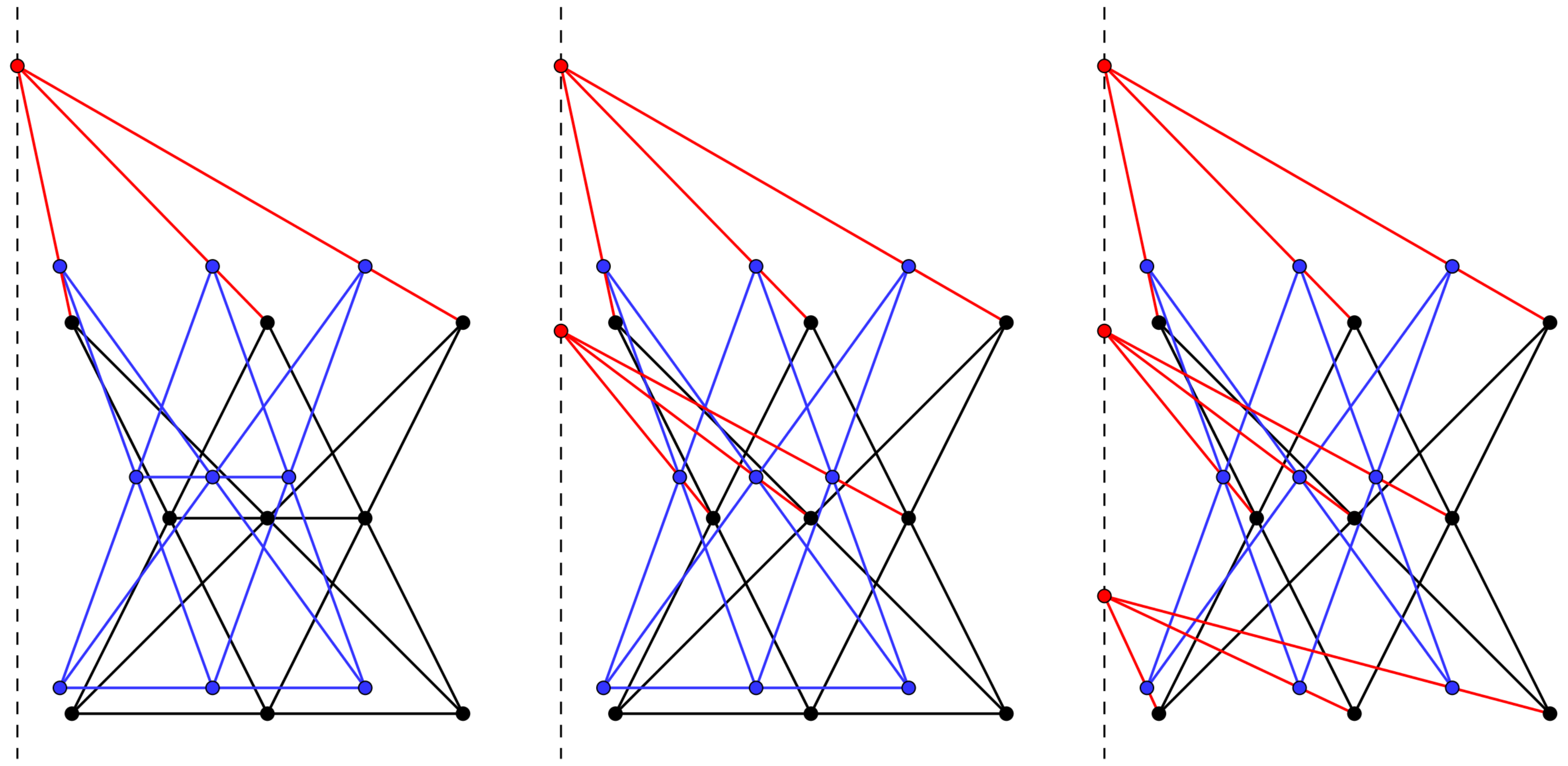
$(m_{k-1}) \to ((k+1)m_k)$ with m parallel lines



Pappus configuration $(9_3) \to (45_4)$ with 9 parallel lines

Affine Switch

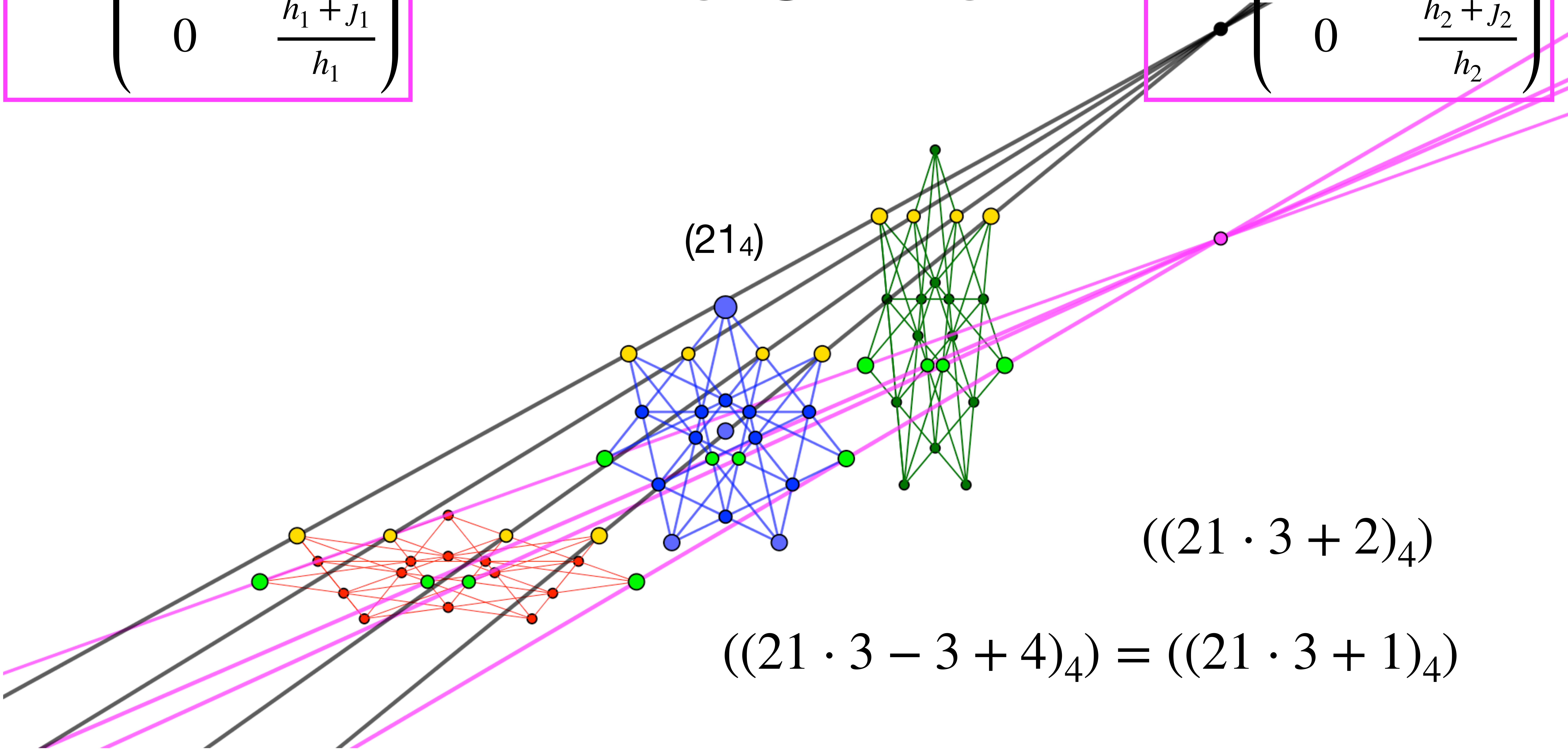
Construct a “band” $((k-1)m+1)_k$, …, $((k-1)m+p)_k$ of consecutive k-configurations from an initial (m_k) configuration with parallel lines



Affine Switch

$$M_1 = \begin{pmatrix} \frac{h_1 - j_1}{h_1} & 0 \\ 0 & \frac{h_1 + j_1}{h_1} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{h_2 - j_2}{h_2} & 0 \\ 0 & \frac{h_2 + j_2}{h_2} \end{pmatrix}$$



$$((21 \cdot 3 + 2)_4)$$

$$((21 \cdot 3 - 3 + 4)_4) = ((21 \cdot 3 + 1)_4)$$

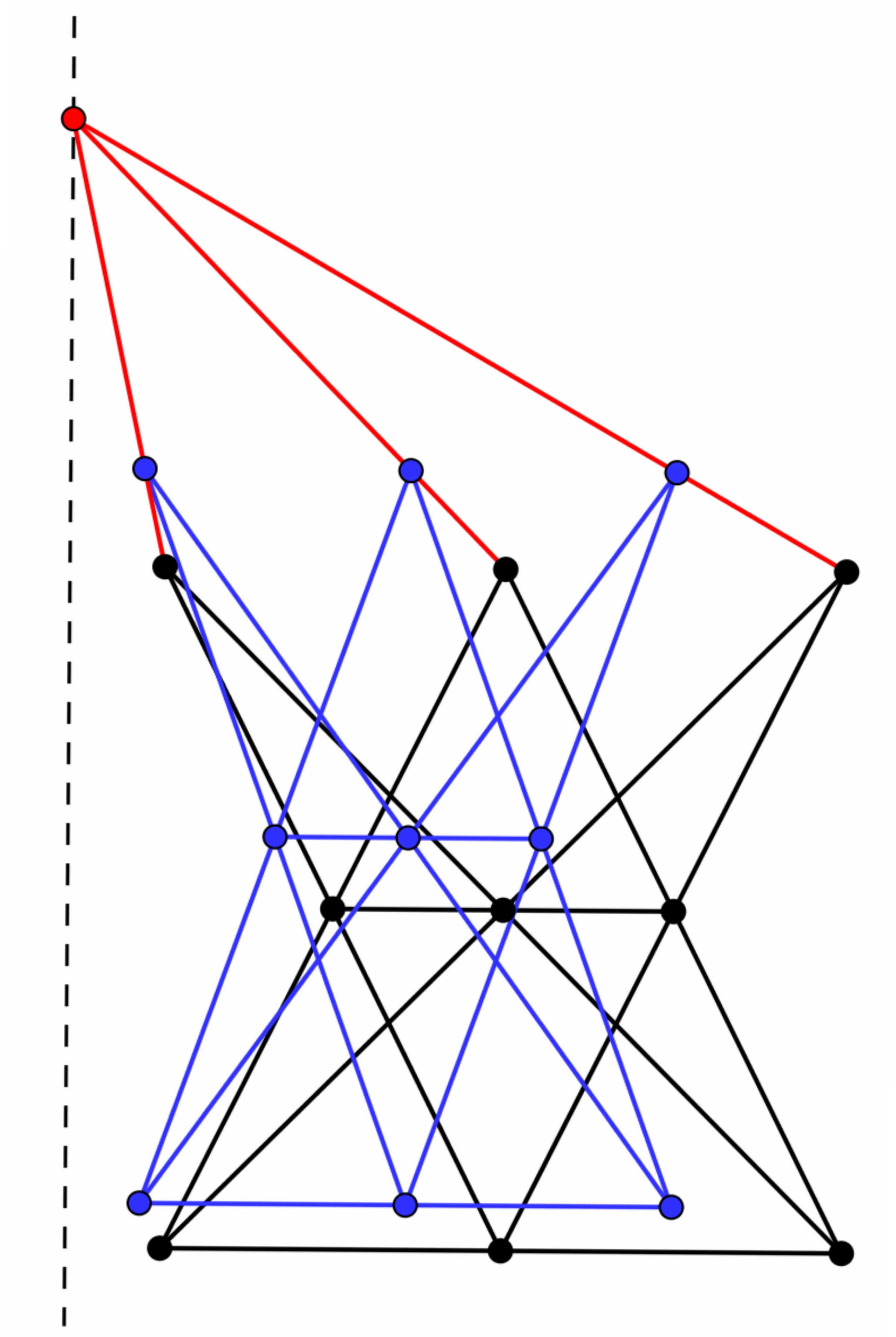
Affine Switch

- Using k-1 affine transformations of the form

$$M_{h,j} = \begin{pmatrix} \frac{h-j}{h} & 0 \\ 0 & \frac{h+j}{h} \end{pmatrix}$$

  corresponding image points are collinear, and when (deleted) configuration lines are parallel to the x-axis, the corresponding lines intersect at a single point on the y-axis!

- Given (m_k) with p *configuration*-disjoint lines parallel to x-axis, can **simultaneously** construct
  $((k\text{-}1)m+1)_k$, …, $((k\text{-}1)m+p)_k$



Constructing consecutive configurations

Consecutive configurations (a_k), $((a+1)_k)$, …, (b_k): abbreviate $[a:b]_k$

(X_{k-1}) **Affine Replication** $((k+1)X_k)$ with X parallel lines **Affine switch** ~~$[\,((k-1)(k+1)X+1 : ((k-1)(k+1)X+X)]_k$~~

$[\,((k-1)(k+1)X : ((k-1)(k+1)X+X)]_k$

$((X+1)_{k-1})$ **Affine Replication** $((k+1)(X+1)_k)$ with $X+1$ parallel lines **Affine switch** $[\,((k-1)(k+1)(X+1) : ((k-1)(k+1)(X+1)+X+1)]_k$

Consecutive when $X \geq k^2 - 2$

Easy induction…

Theorem (B., Gévay, Pisanski 2021):
For any $k \geq 2$, the number N_k exists.

Proof:
Base case: $N_2 = 3$.
Induction Hypothesis: N_{k-1} exists.
Inductive step: $N_k \leq (k^2 - 1) \max(N_{k-1}, k^2 - 2)$.

Bounds

k	$\bar{N}_k$ with $N_2^R = 3$	$\bar{N}_k$ with $N_3^R = 9$	$\bar{N}_k$ with $N_4^R = 24$	N_k^R
2	**3**	-	-	**3**
3	56	**9**	-	**9**
4	840	210	**24**	**24**
5	20 160	5 040	**576**	**576**
6	705 600	176 400	**20 160**	**20 160**
7	33 868 800	8 467 200	**967 680**	**967 680**
8	2 133 734 400	533 433 600	**60 963 840**	**60 963 840**
9	170 698 752 000	42 674 688 000	**4 877 107 200**	**4 877 107 200**
10	16 899 176 448 000	4 224 794 112 000	**482 833 612 800**	**482 833 612 800**

N_k^R is the current record; $\bar{N}_k = (k^2 - 1)\max(N_{k-1}^R, k^2 - 2)$

Can we do better?

t-configurations $a, a + d, a + 2d, \ldots, X, \ldots$

Affine Replication

(t+1) configurations $(t + 2)a, (t + 2)(a + d), (t + 2)(a + 2d), \ldots$

Affine Replication

(t+2)-configurations $(t + 3)(t + 2)a, (t + 3)(t + 2)(a + d), (t + 3)(t + 2)(a + 2d), \ldots$

…

Affine Replication

k-configurations $\frac{(k+1)!}{(t+1)!}a, \frac{(k+1)!}{(t+1)!}(a + d), \frac{(k+1)!}{(t+1)!}(a + 2d), \ldots$

Starting t-configuration X produces k-configuration with $\frac{k!}{(t+1)!}X$ parallel lines

Can we do better?

t-configurations $a, a+d, a+2d, \ldots, X, \ldots$

...

***k*-configurations** $\frac{(k+1)!}{(t+1)!}a, \frac{(k+1)!}{(t+1)!}(a+d), \frac{(k+1)!}{(t+1)!}(a+2d), \ldots$

Affine Switch

$$\left[(k-1)\frac{(k+1)!}{(t+1)!}X : (k-1)\frac{(k+1)!}{(t+1)!}X + \frac{k!}{(t+1)!}X\right],$$

$$\left[(k-1)\frac{(k+1)!}{(t+1)!}(X+d) : (k-1)\frac{(k+1)!}{(t+1)!}(X+d) + \frac{k!}{(t+1)!}(X+d)\right]\ldots$$

Overlap when $X \geq (k^2-1)d - \frac{(t+1)!}{k!}$

New Bounds

Old bounds: $N_5 \leq 576$, $N_6 \leq 20160$

Recursively define $\hat{N}_k = (k^2 - 1) \min_{3 \leq t \leq k-1} \left\{ \frac{k!}{(t+1)!} \max\left\{\hat{N}_t, k^2 - 1\right\} \right\}$

with $\hat{N}_3 = 9$, $\hat{N}_4 = 24$

$$\hat{N}_5 = (5^2 - 1) \min_{3 \leq t \leq 4} \left\{ \frac{5!}{(t+1)!} \max\left\{\hat{N}_t, 5^2 - 1\right\} \right\} = 24 \min \left\{ \underbrace{\frac{5!}{4!} \max\{9,24\}}_{t=3}, \underbrace{\frac{5!}{5!} \max\{24,24\}}_{t=4} \right\} = \mathbf{576}$$

$$\hat{N}_6 = (6^2 - 1) \min_{3 \leq t \leq 5} \left\{ \frac{6!}{(t+1)!} \max\left\{\hat{N}_t, 6^2 - 1\right\} \right\} = 35 \min \left\{ \underbrace{\frac{6!}{4!} \max\{9,35\}}_{t=3}, \underbrace{\frac{6!}{5!} \max\{24,35\}}_{t=4}, \underbrace{\frac{6!}{6!} \max\{576,35\}}_{t=5} \right\} = 35(35 \cdot 6) = \mathbf{7350}$$

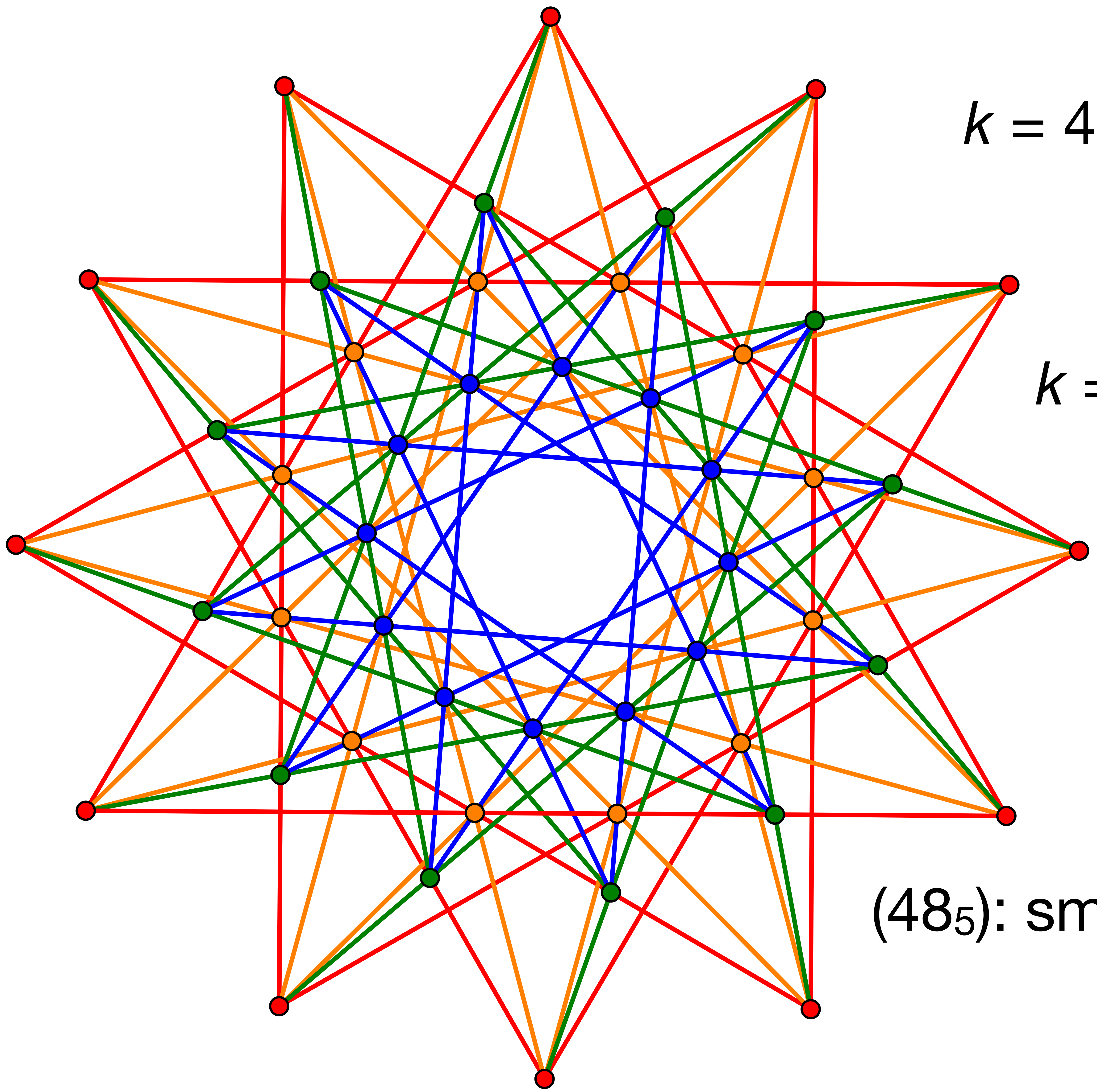
...

$$\hat{N}_{10} = 99 \min \left\{ \underbrace{\frac{10!}{4!} \max\{9,99\}}_{t=3}, \underbrace{\frac{10!}{5!} \max\{24,99\}}_{t=4}, \underbrace{\frac{10!}{6!} \max\{576,99\}}_{t=5}, \frac{10!}{7!} \max\{7350,99\}, \ldots, \frac{10!}{10!} \max\{\hat{N}_9,99\} \right\} = \frac{10!}{6!} \cdot 576 \cdot 99 = 287400960$$

New bounds

k	$\hat{N}_k = N_k^R$	formula	initial sequence
4	**24**	-	-
5	**576**	$(5^2-1)^2$	$t=4$
6	7 350	$6(6^2-1)^2$	$t=4$
7	96 768	$7\cdot 6\cdot(7^2-1)^2$	$t=4$
8	1 333 584	$\frac{8!}{5!}(8^2-1)^2$	$t=4$
9	19 353 600	$\frac{9!}{5!}(9^2-1)^2$	$t=4$
10	287 400 960	$\frac{10!}{6!}\cdot\mathbf{576}\cdot(10^2-1)$	$\mathbf{t=5}$
11	3 832 012 800	$\frac{11!}{6!}\cdot 576\cdot(11^2-1)$	$t=5$
$\vdots$			
24	$\approx 2.85\times 10^{26}$	$\frac{24!}{6!}\cdot 576\cdot(24^2-1)$	$t=5$
25	$\approx 8.39\times 10^{27}$	$\frac{25!}{6!}\cdot\mathbf{(25^2-1)^2}$	$t=5$
26	$\approx 8.02\times 10^{30}$	$\frac{26!}{6!}\cdot(26^2-1)^2$	$t=5$
$\vdots$			
32	$\approx 3.82\times 10^{38}$	$\frac{32!}{6!}\cdot(32^2-1)^2$	$t=5$
33	$\approx 1.38\times 10^{40}$	$\frac{33!}{7!}\cdot\mathbf{7350}\cdot(33^2-1)$	$\mathbf{t=6}$
$\vdots$			

How can we do better?



k = 4*:* does there exist a (23_4) configuration?

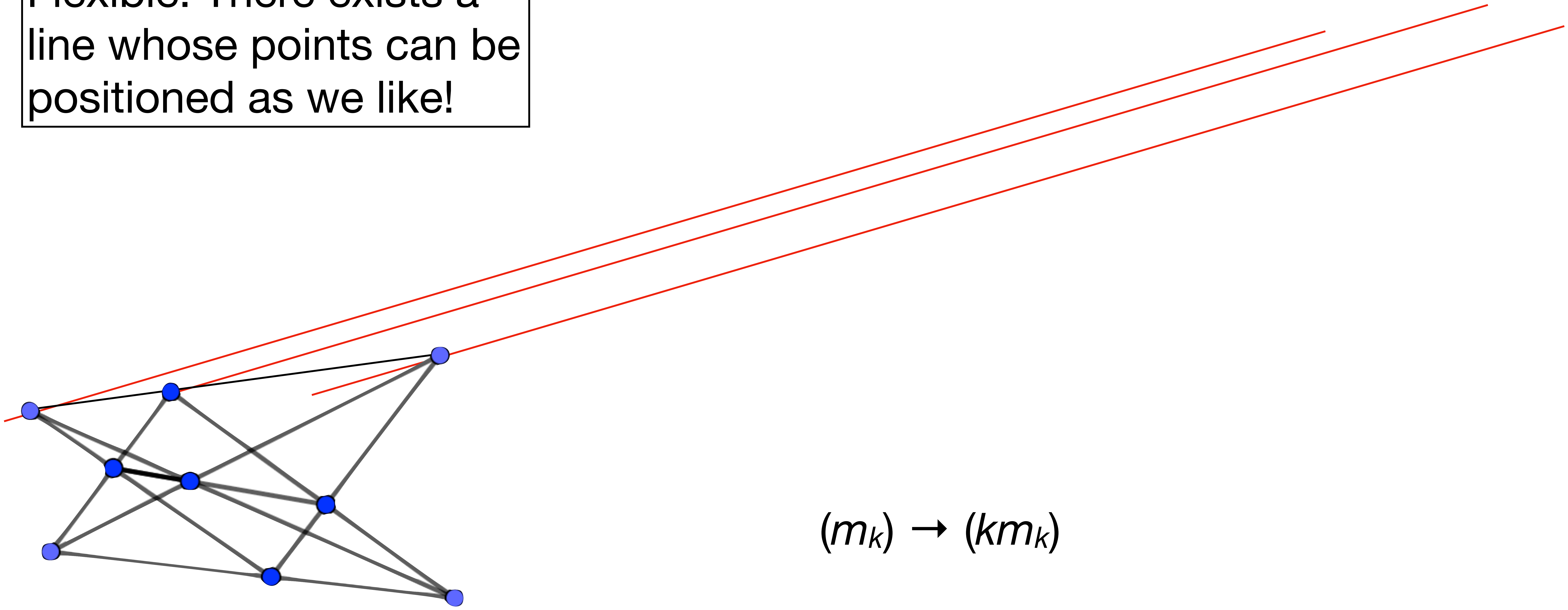
k = 5*:* Ad hoc constructions and additional Grünbaum Calculus constructions!

(48_5): smallest known 5-configuration

More Grünbaum Calculus operations

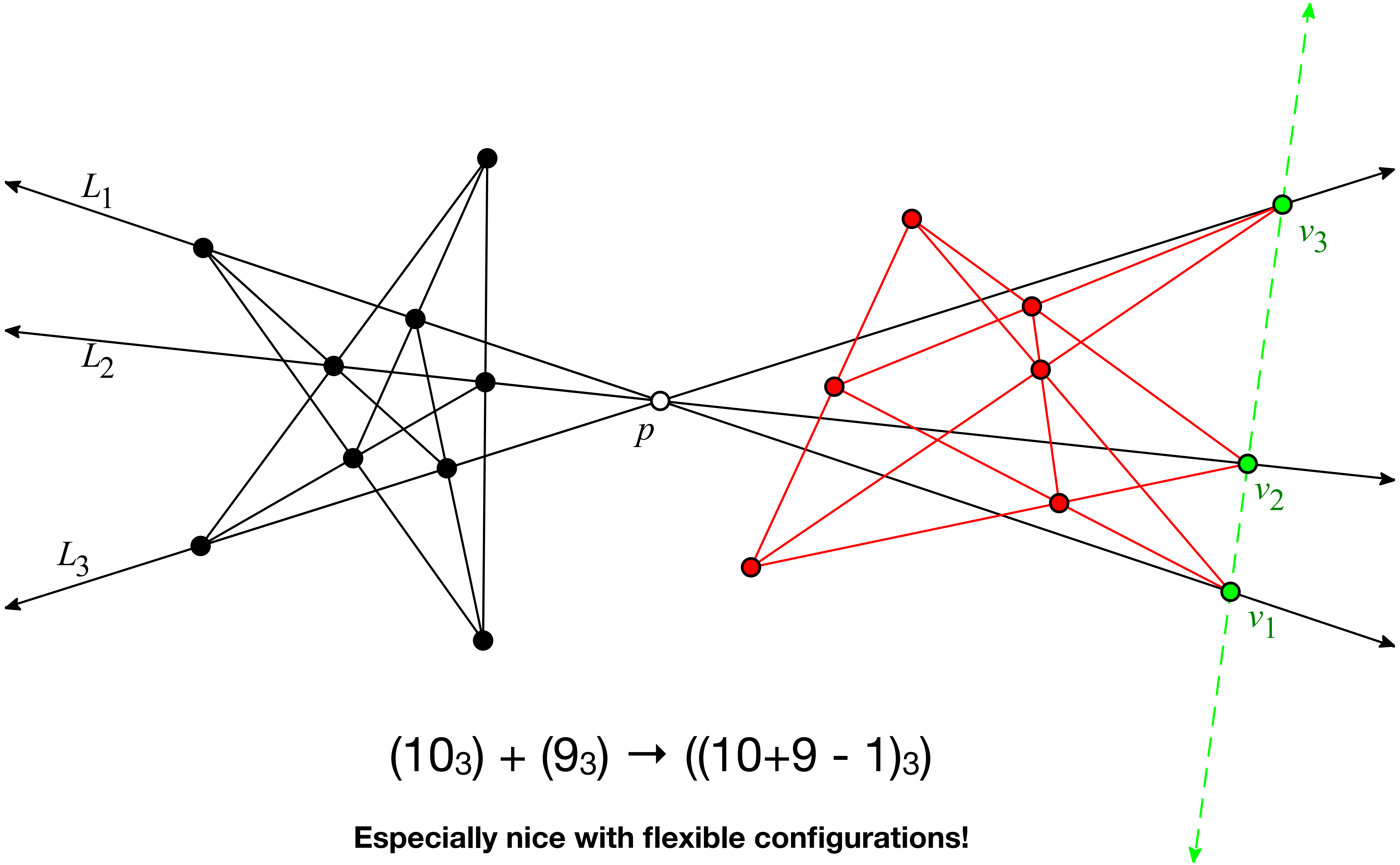
Parallel Switch

Flexible! There exists a line whose points can be positioned as we like!



$(m_k) \to (km_k)$

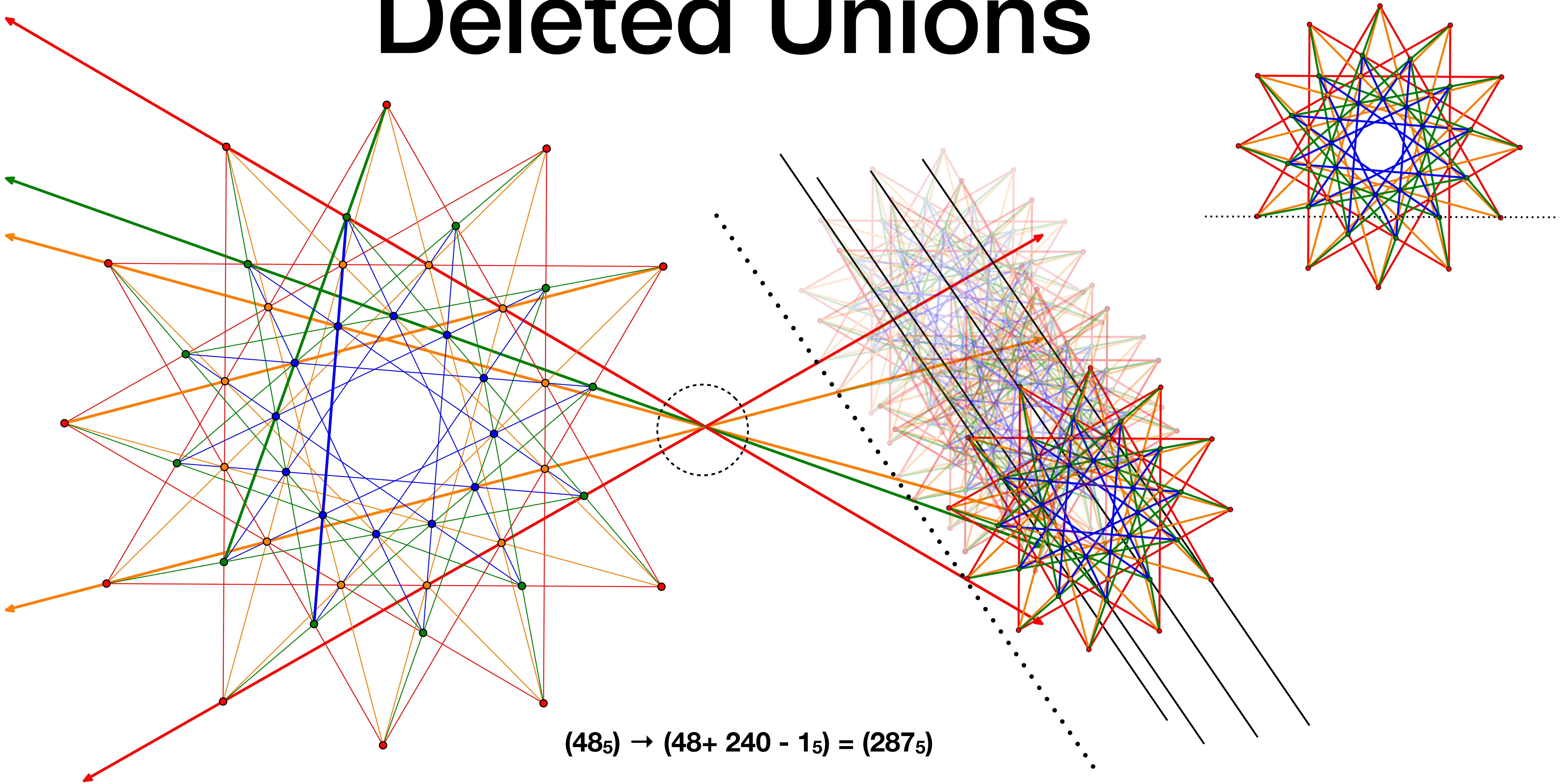
Deleted Unions



$(10_3) + (9_3) \rightarrow ((10+9 - 1)_3)$

Especially nice with flexible configurations!

Deleted Unions

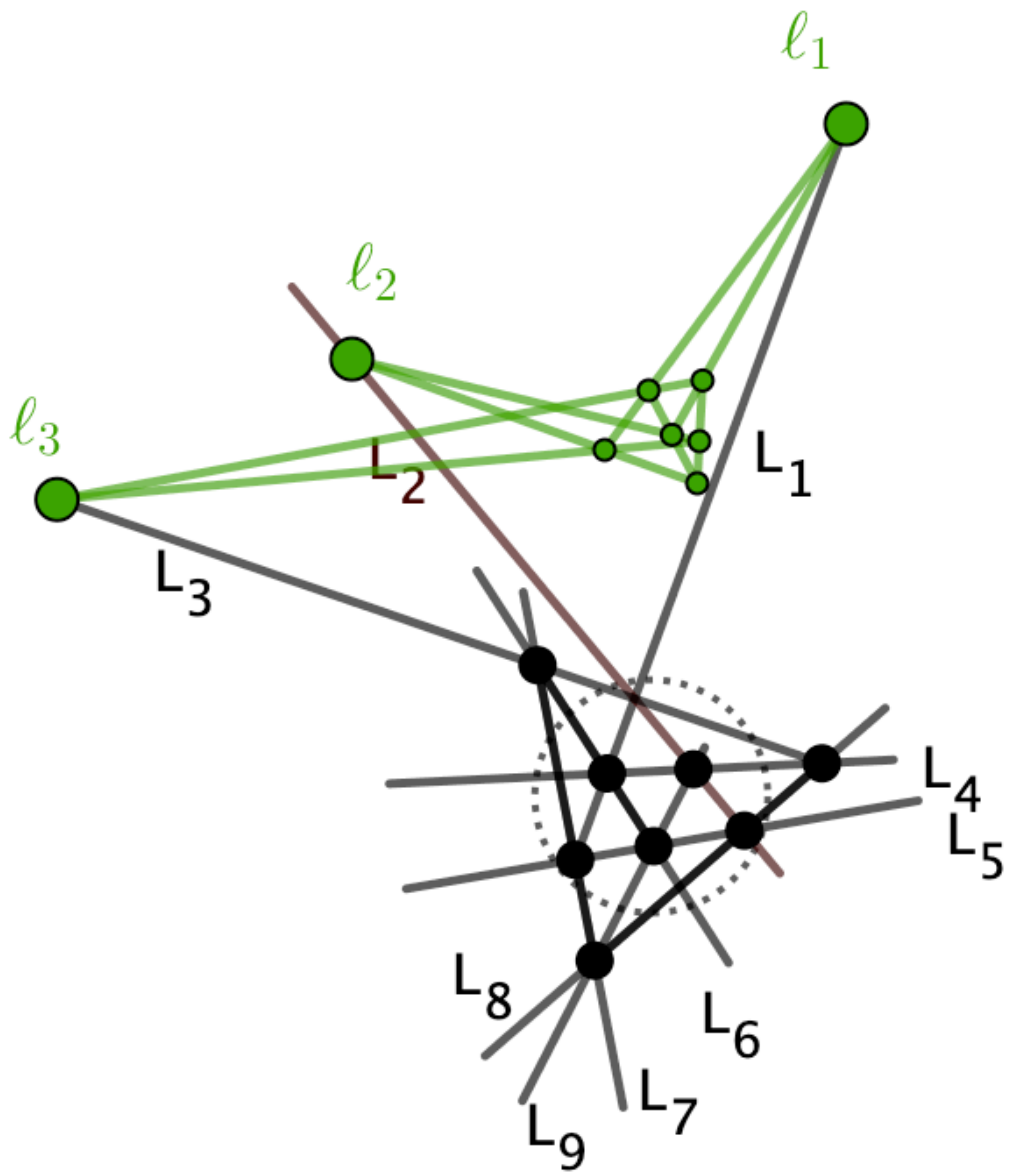


$(48_5) \rightarrow (48 + 240 - 1_5) = (287_5)$

Especially nice with flexible configurations — combine with parallel switch!

Deleted Unions

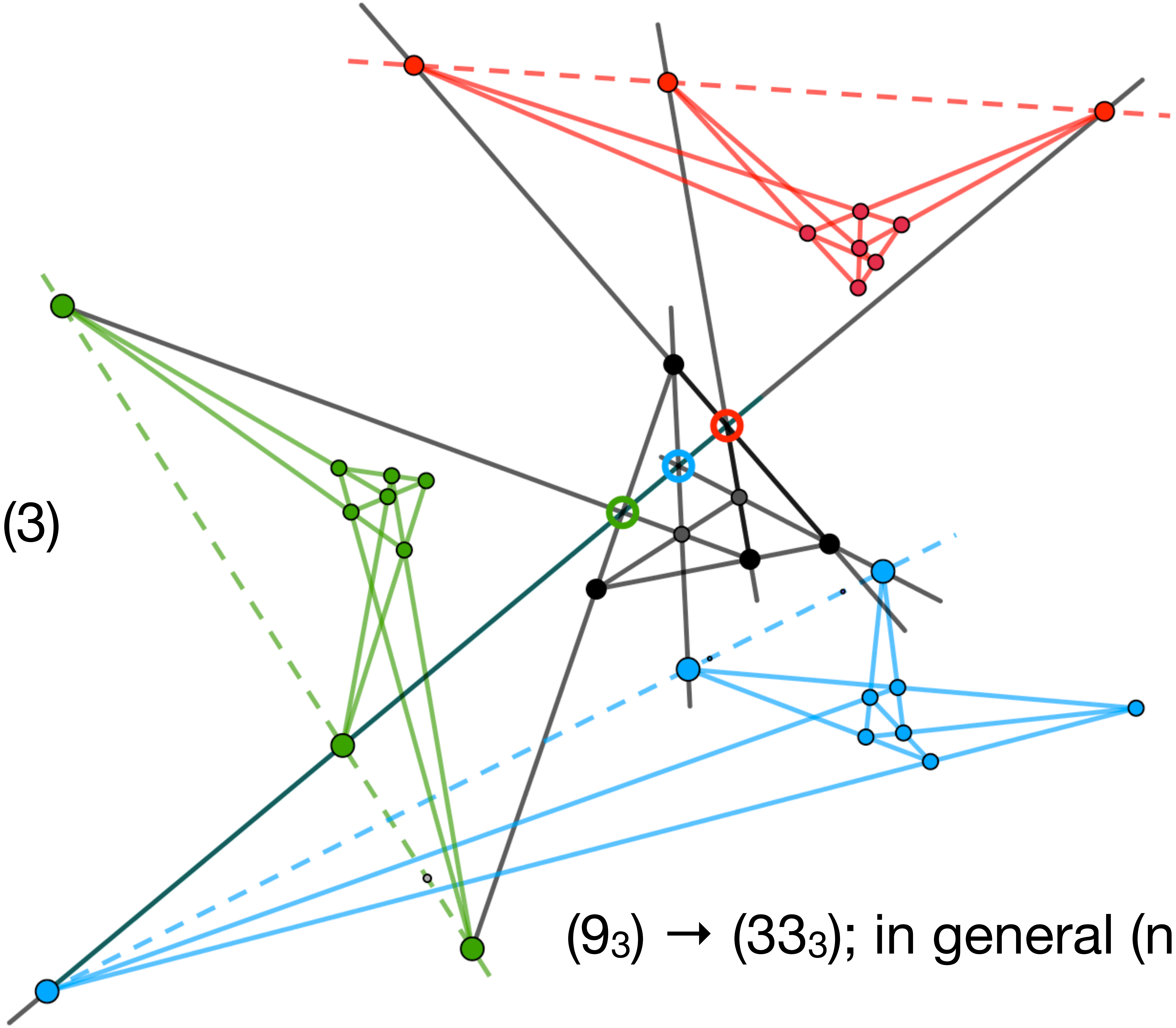
- Construct (n_k) configuration C and a conic
- Choose point P to delete
- Intersect a line ℓ through the lines L_1, …, L_k through P to form $\ell_1, \ldots, \ell_k$
- Construct polar C^* with images v_1, …, v_k of L_1, …, L_k
- Find a collineation to map the image points v_1,v_2,v_3 to ℓ_1, ℓ_2, ℓ_3 and apply the collineation to C^*; projective geometry says $v_i \mapsto \ell_i$
- Delete the polar and the point P, line ℓ



(9_3) → (17_3); in general (n_k) → ($2n$-1_k)

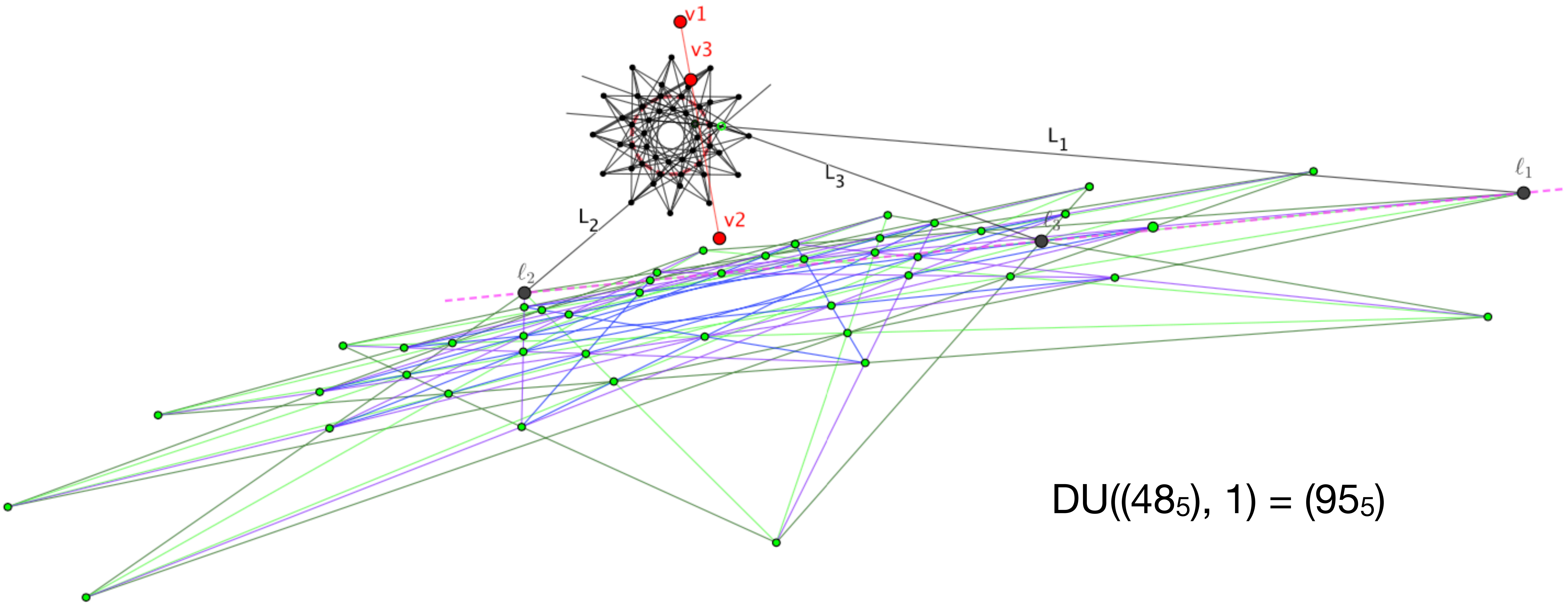
Deleted Unions

DU(3)



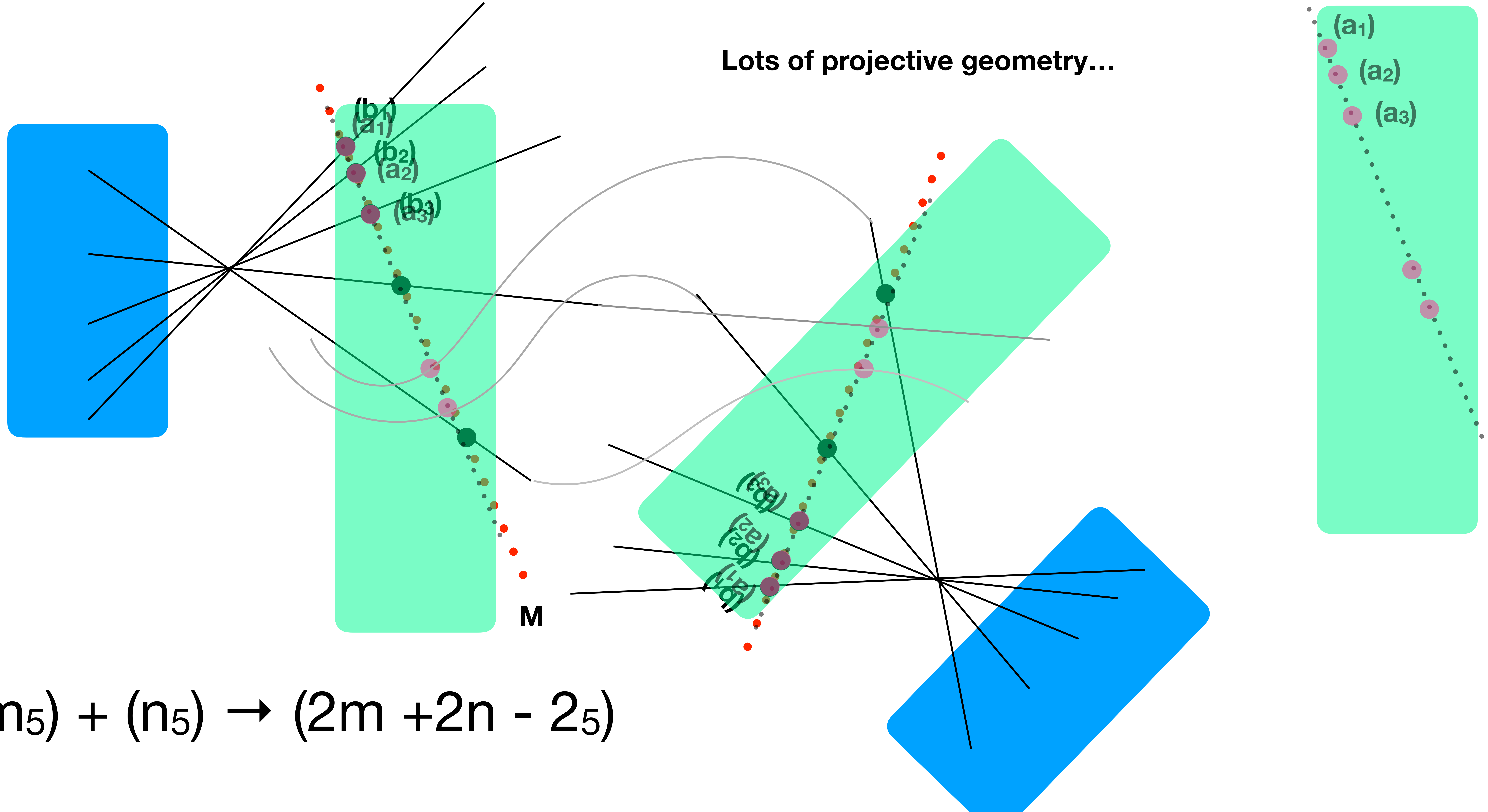
$(9_3) \to (33_3)$; in general $(n_k) \to (((t+1)n - t)_k)$

Deleted Unions



$DU((48_5), 1) = (95_5)$

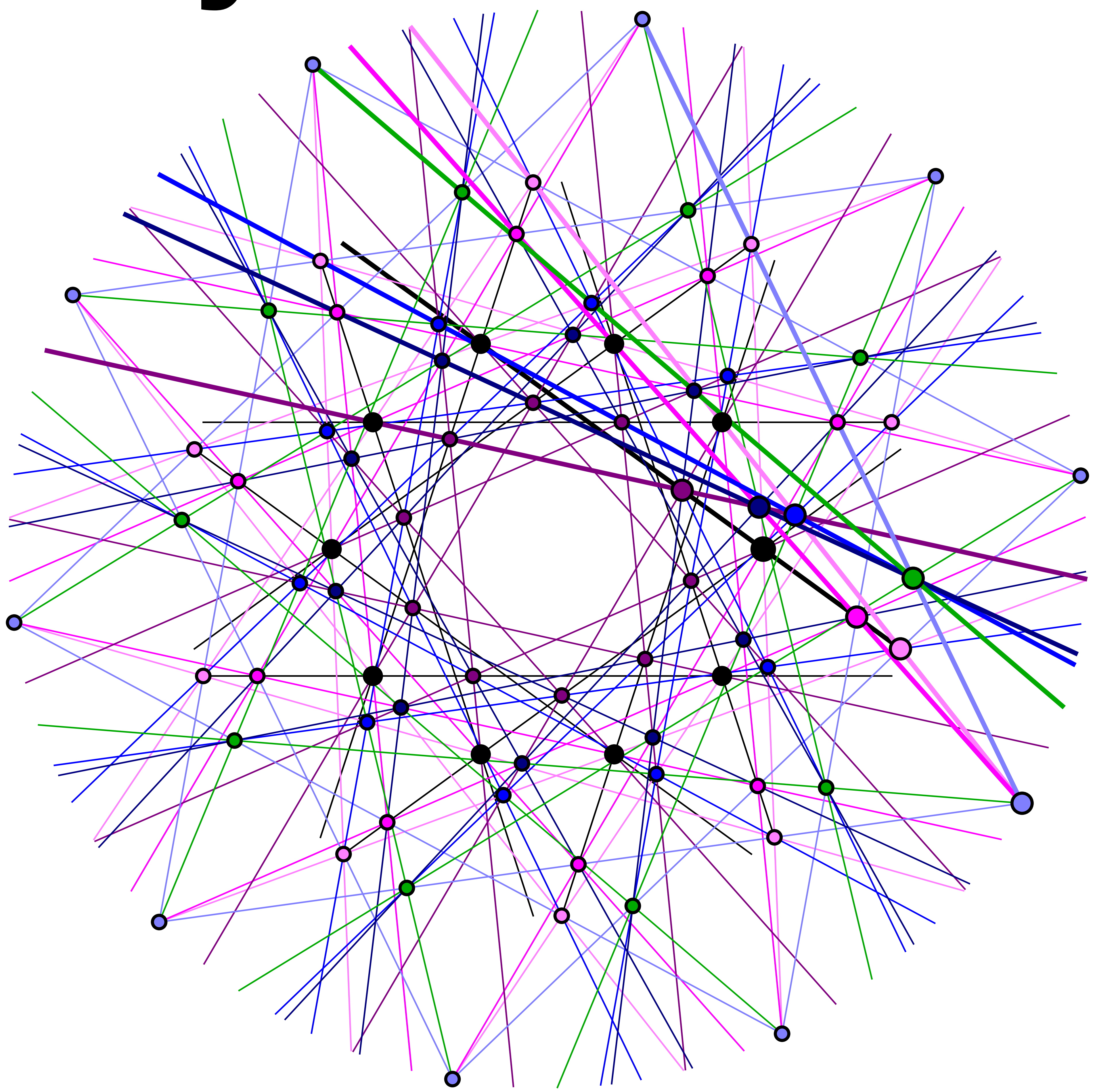
Distributed Deleted Unions

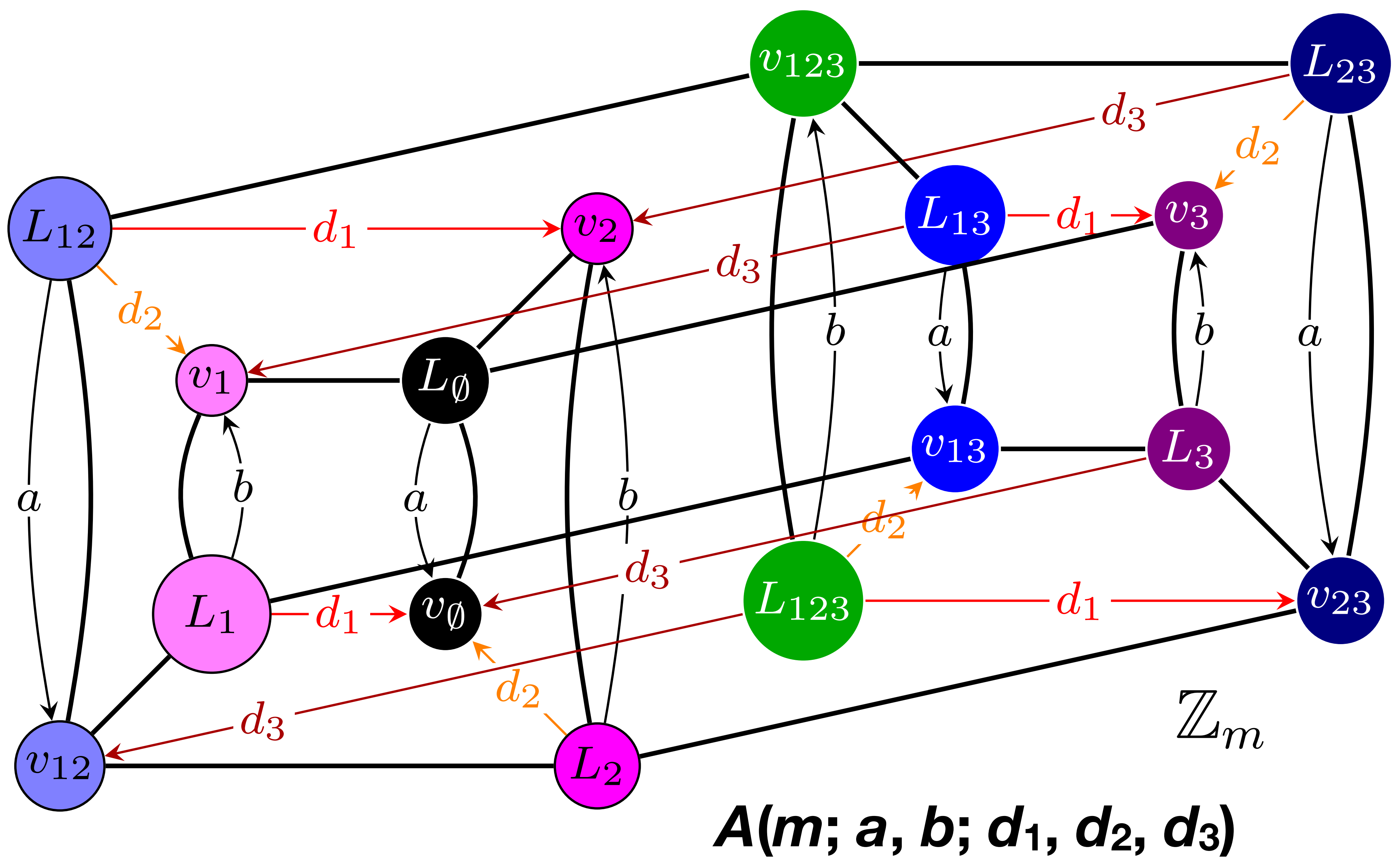


$(m_5) + (n_5) \rightarrow (2m + 2n - 2_5)$

Systematic constructions

Systematic 5-configurations

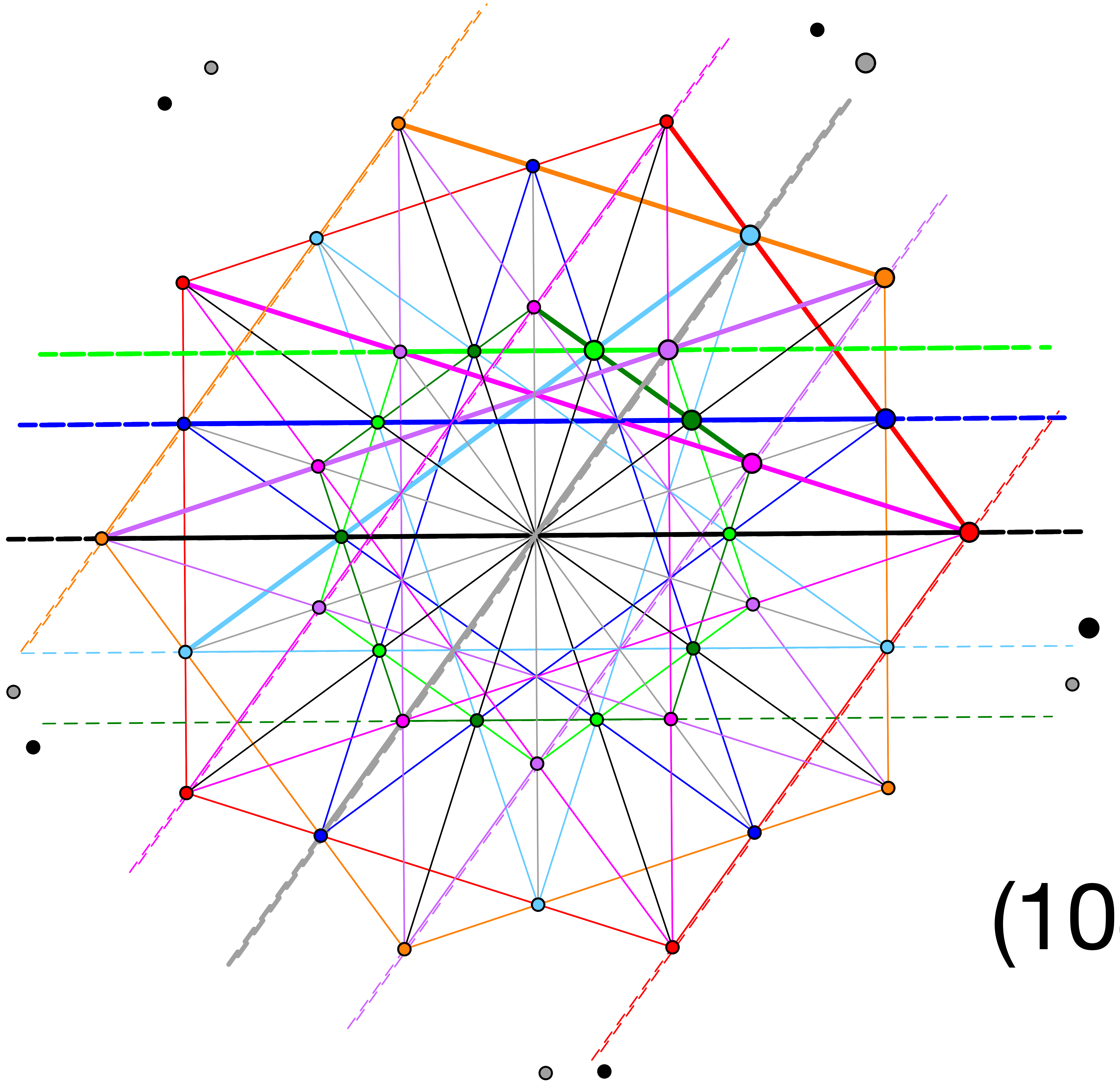




$(8m_5)$: A-series $A(m; 3,3; 1,2,4)$

Systematic 5-configurations

Celestial
$2q\#(2,1;4,3;\ 1,2;3,4)$

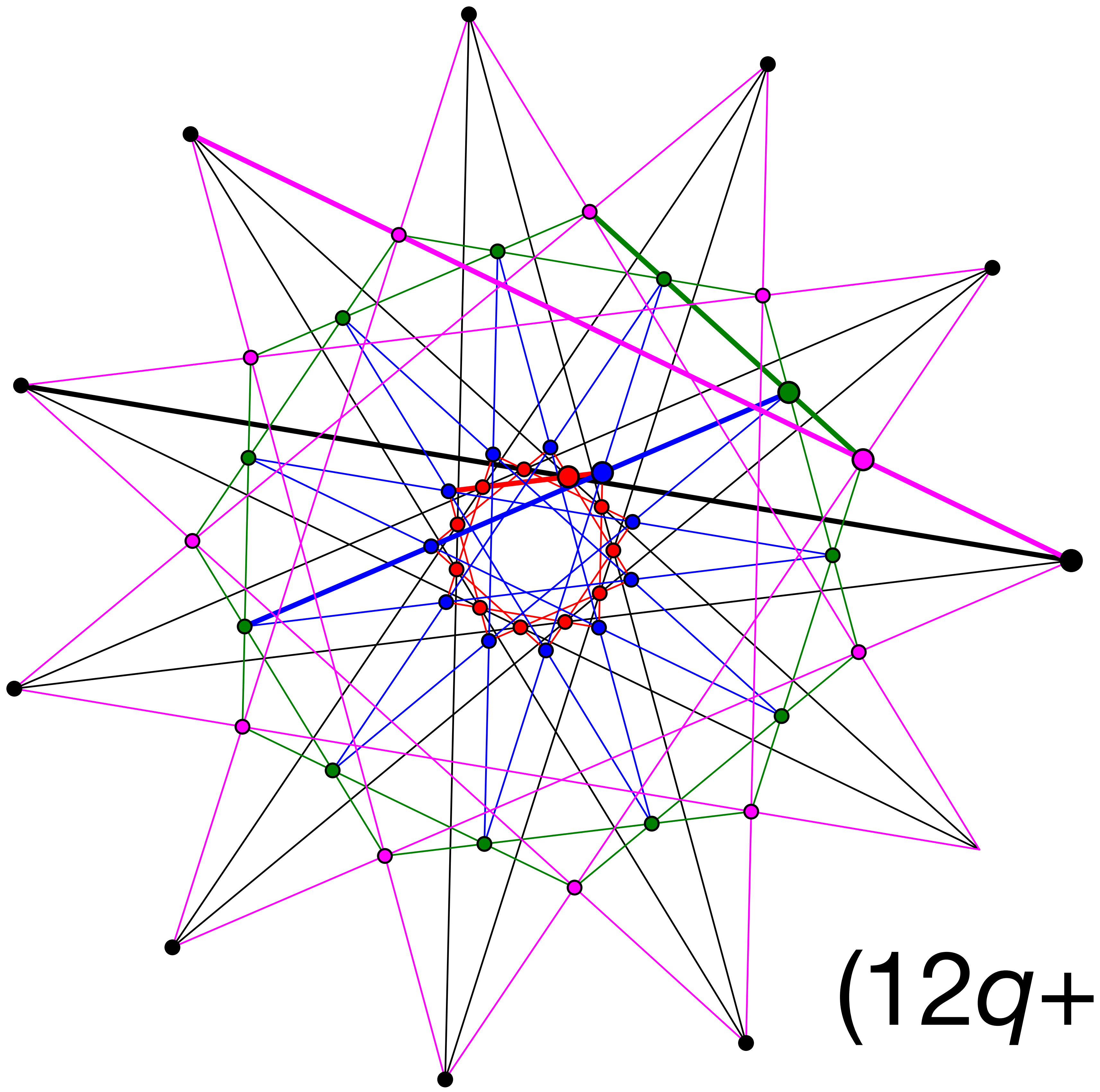


$(10q_5)$ for m ≥ 5

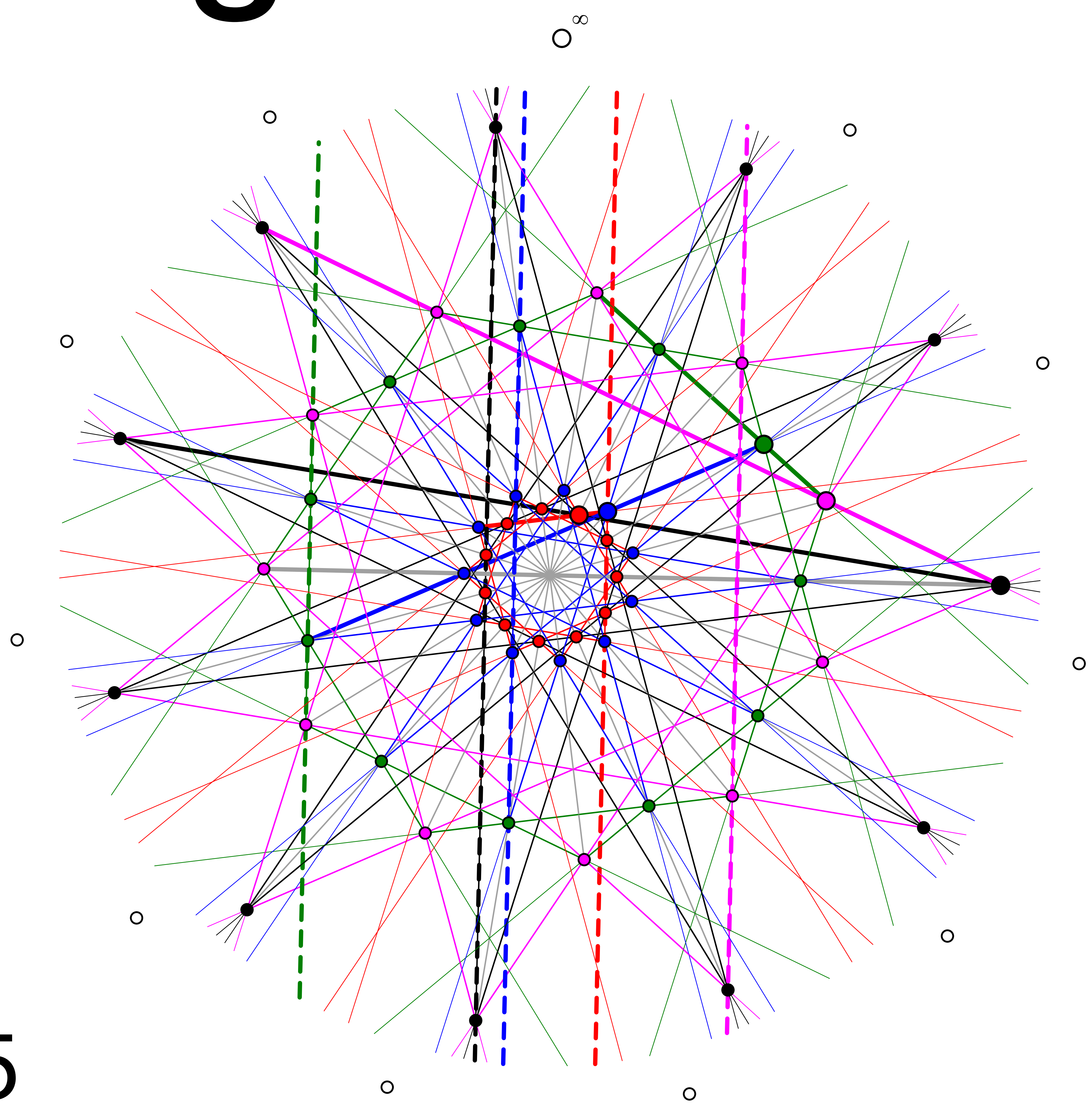
Diameters + ∞ to 4-celestial 4-configurations

Systematic 5-configurations

Celestial (2q+1)#(5,1;2,3;4,5;1,2;3,4)

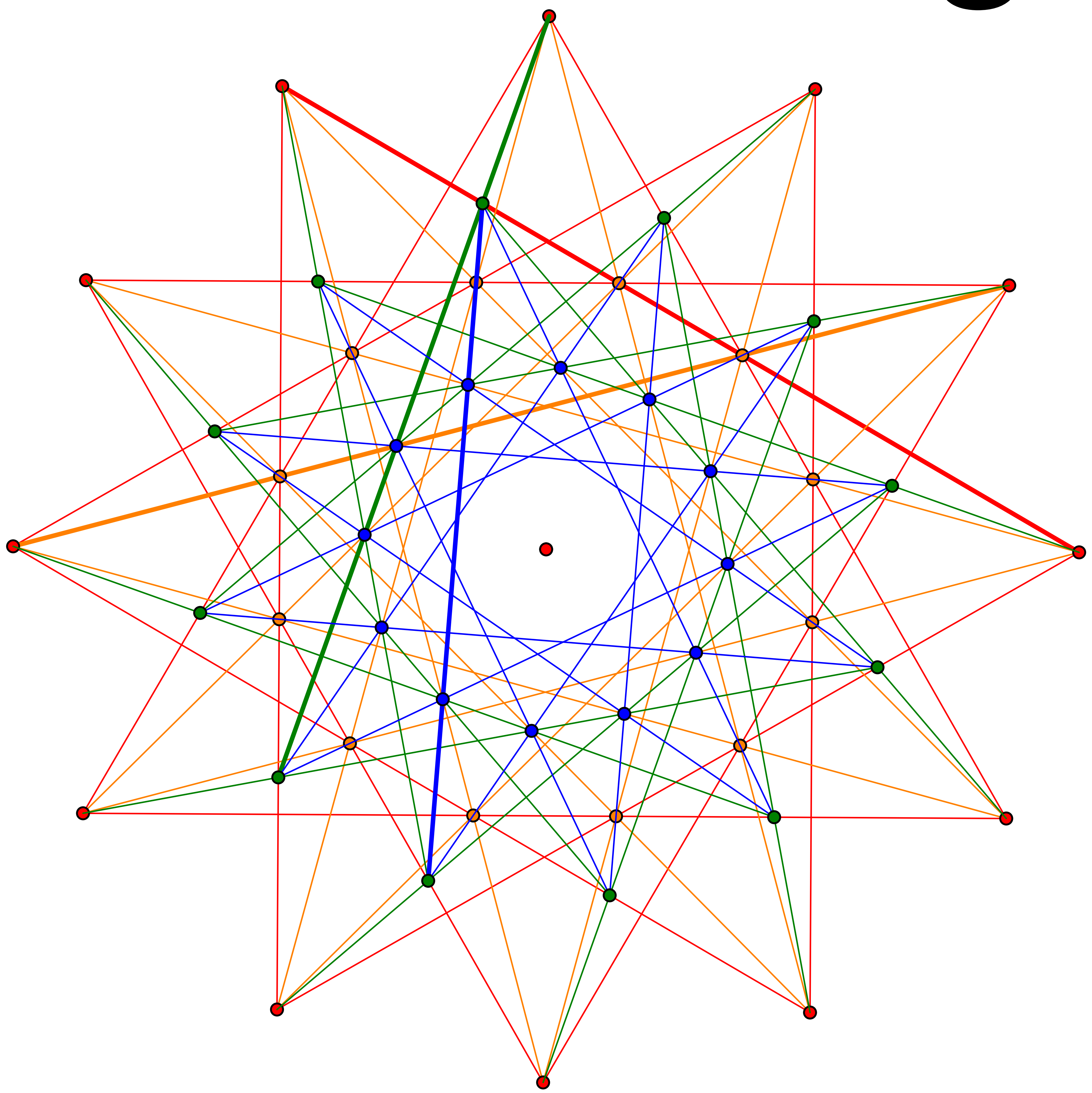


$(12q+6_5)$, $q \geq 5$



Diameters + ∞ to 5-celestial 4-configurations

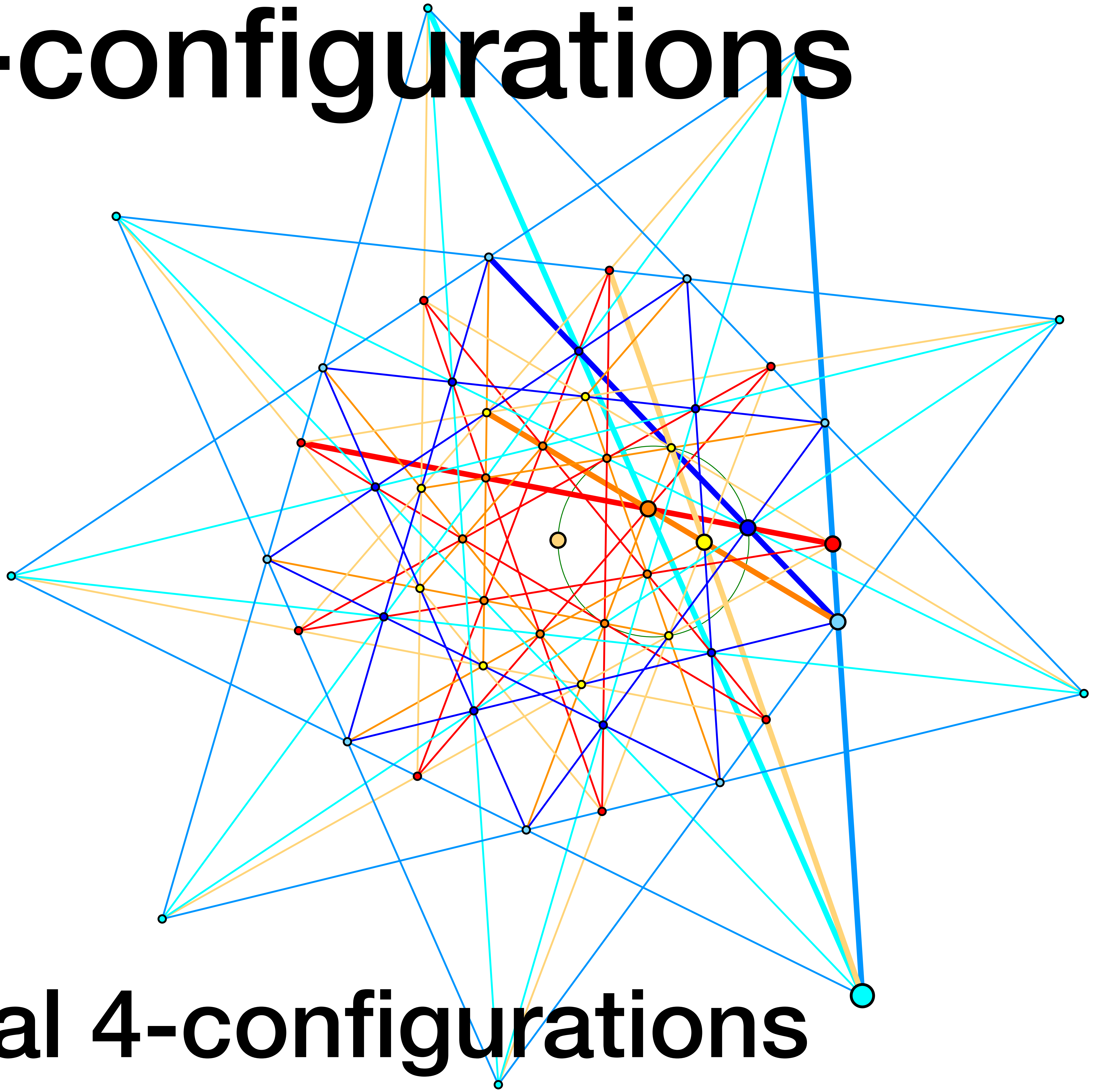
Systematic 5-configurations



“Nesting” celestial 4-configurations

Systematic 5-configurations

- Two Useful Families:
  - $(18q_5)$, $q \geq 5$; smallest is (81_5)
  - $(27q_5)$, $q \geq 3$; smallest is (90_5)
- Two ad-hoc constructions produce (48_5), (54_5)

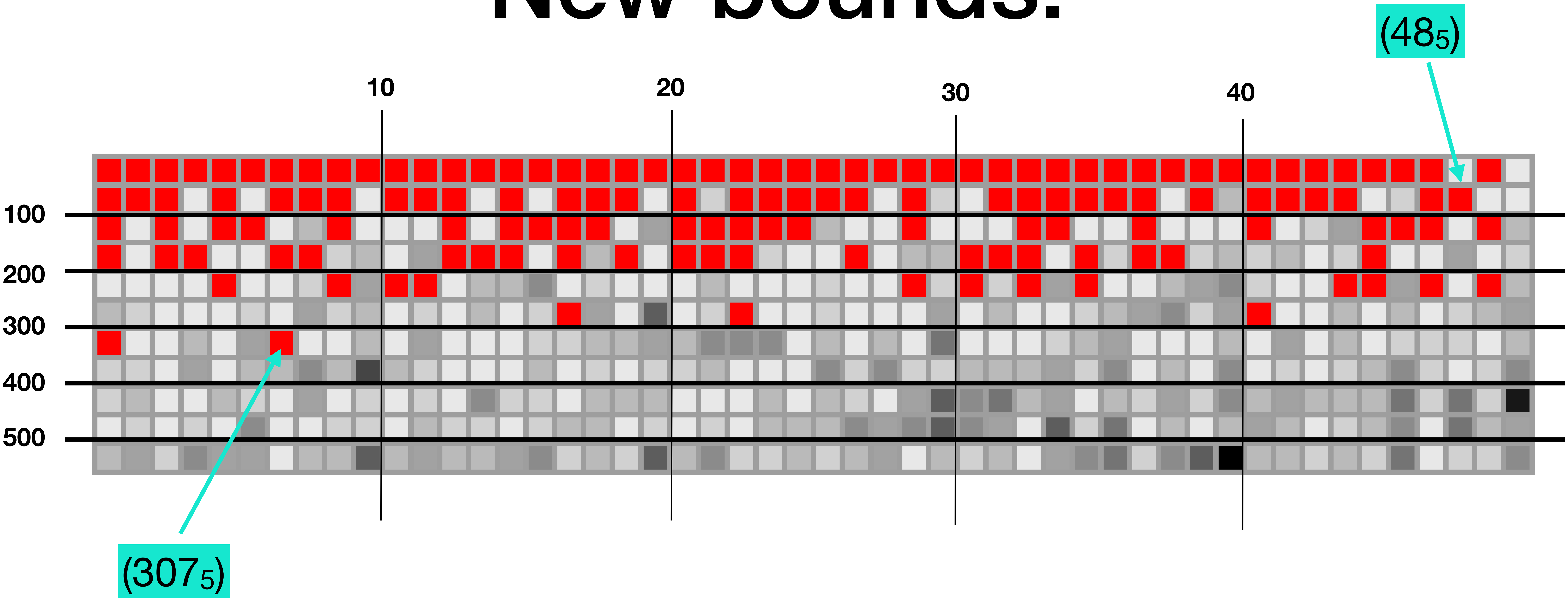


“Nesting” celestial 4-configurations

New bounds!

- Systematic and ad hoc 5-configurations
- Affine Replication of 4-configurations
- Then apply: Parallel Switch, DU(t), DDU, Affine Switch, Parallel Switch, DU(t)…
- Then **look for pairs** to apply DU(C, D) where D is flexible (from Parallel Switch)

New bounds!



Theorem: $N_5 \leq 307$.

What about N_6?

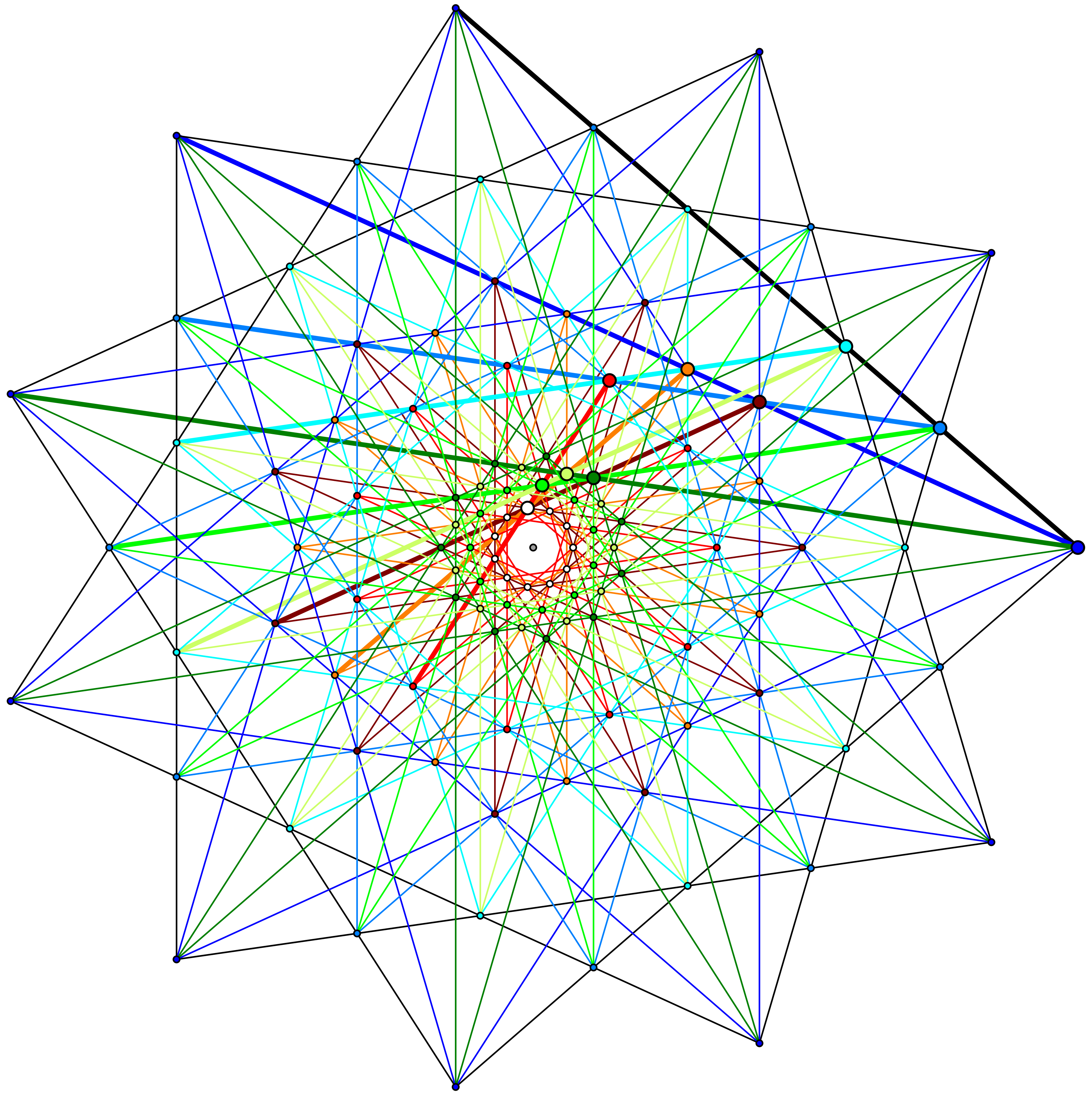
$$\hat{N}_6 = 35 \min\left\{ \frac{6!}{4!}\max\{9,35\}, \frac{6!}{5!}\max\{24,35\}, \frac{6!}{6!}\max\{308,35\} \right\} = 35(35 \cdot 6) = 7350$$

210 308

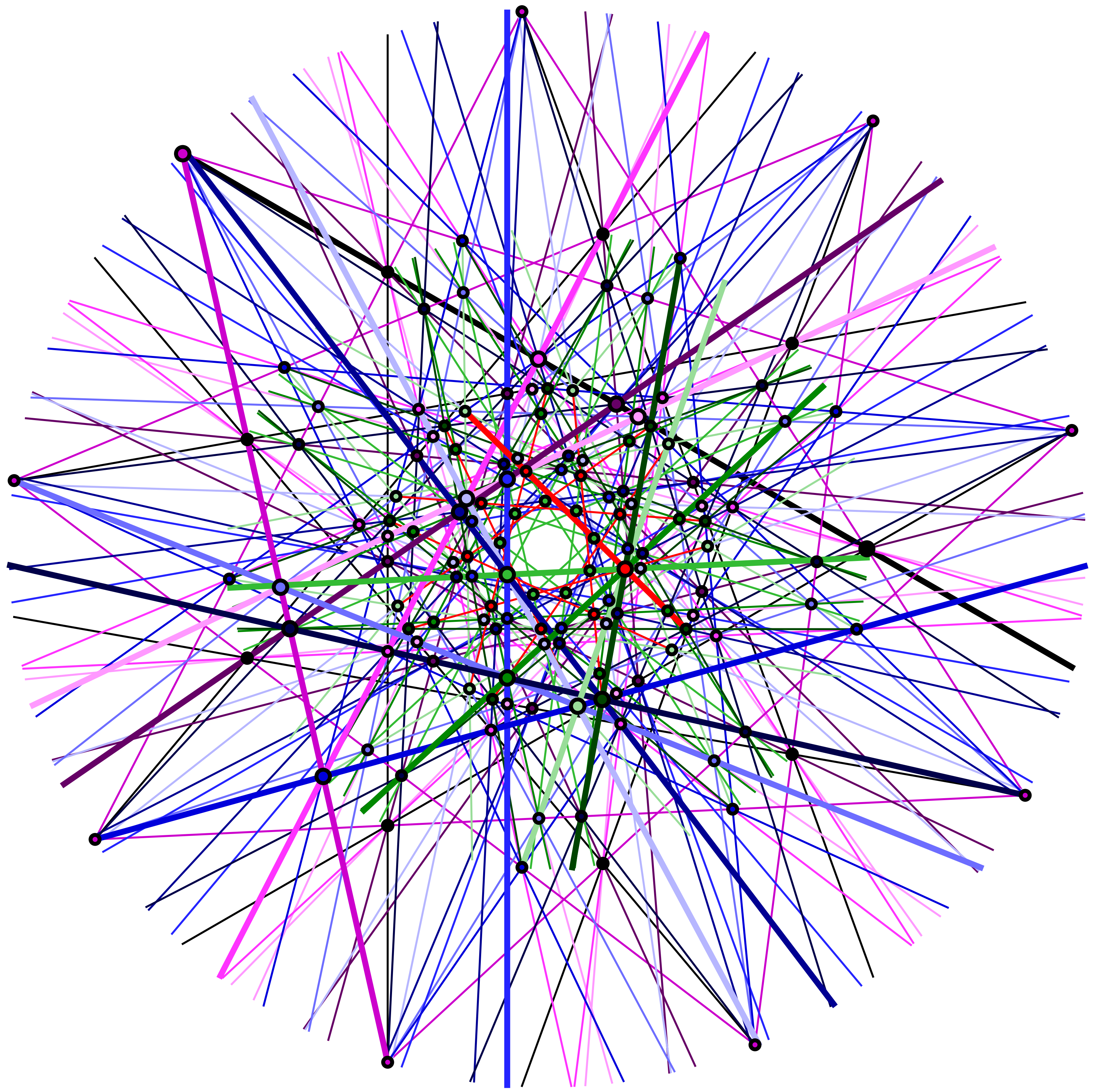
No immediate reduction in bounds…

- Smallest is (96_6)
- Multicelestial $(10m_6)$, $m \geq 11$
- $A(m; 3,3; 1,2,4,5)$: $(16m_6)$, $m \geq 7$
- DU(t), DU(C,D)
- Parallel Switch (flexible!)
- Affine Replication, Switch…

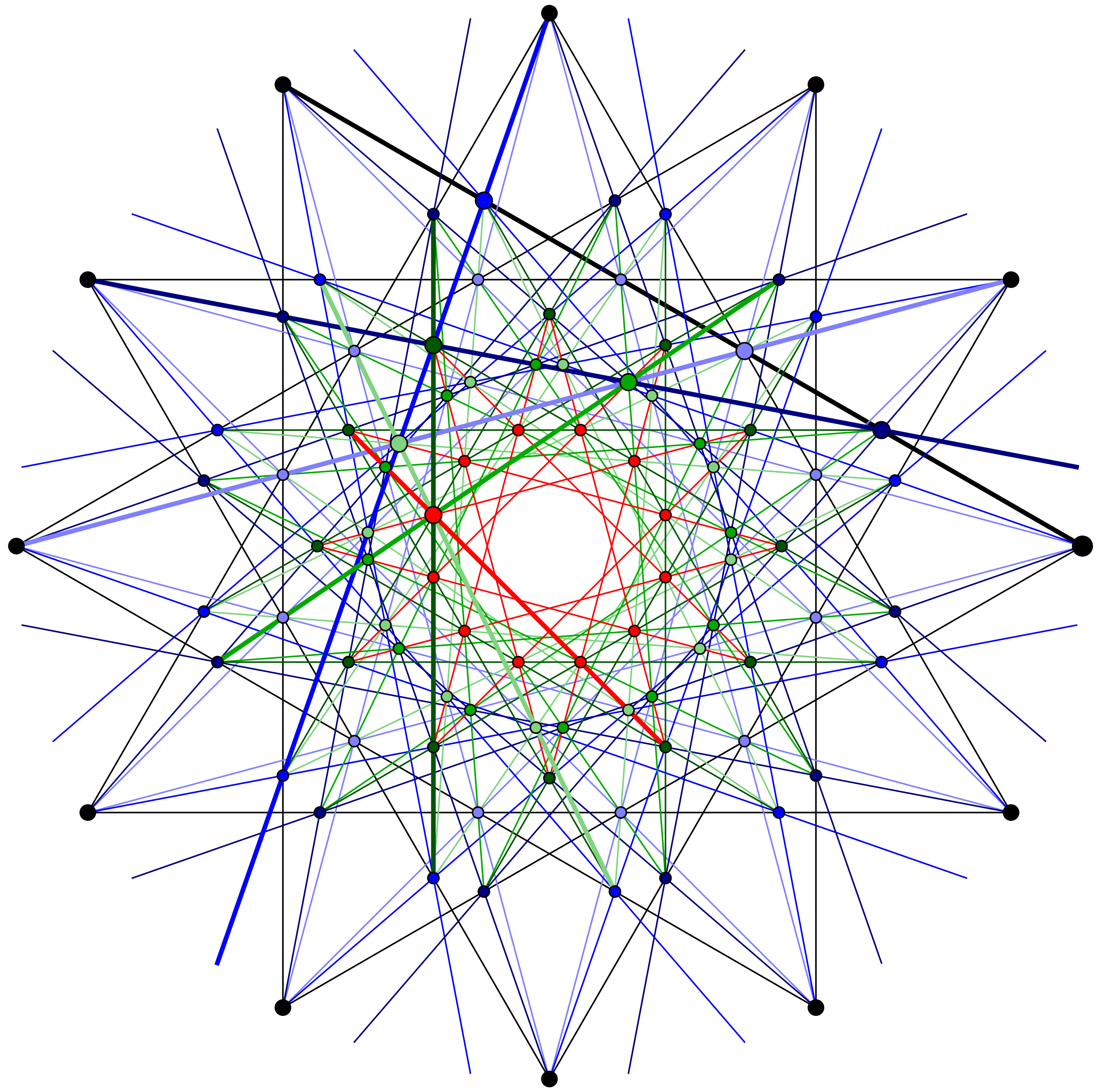
What about N_6?



Multicelestial (110_6)

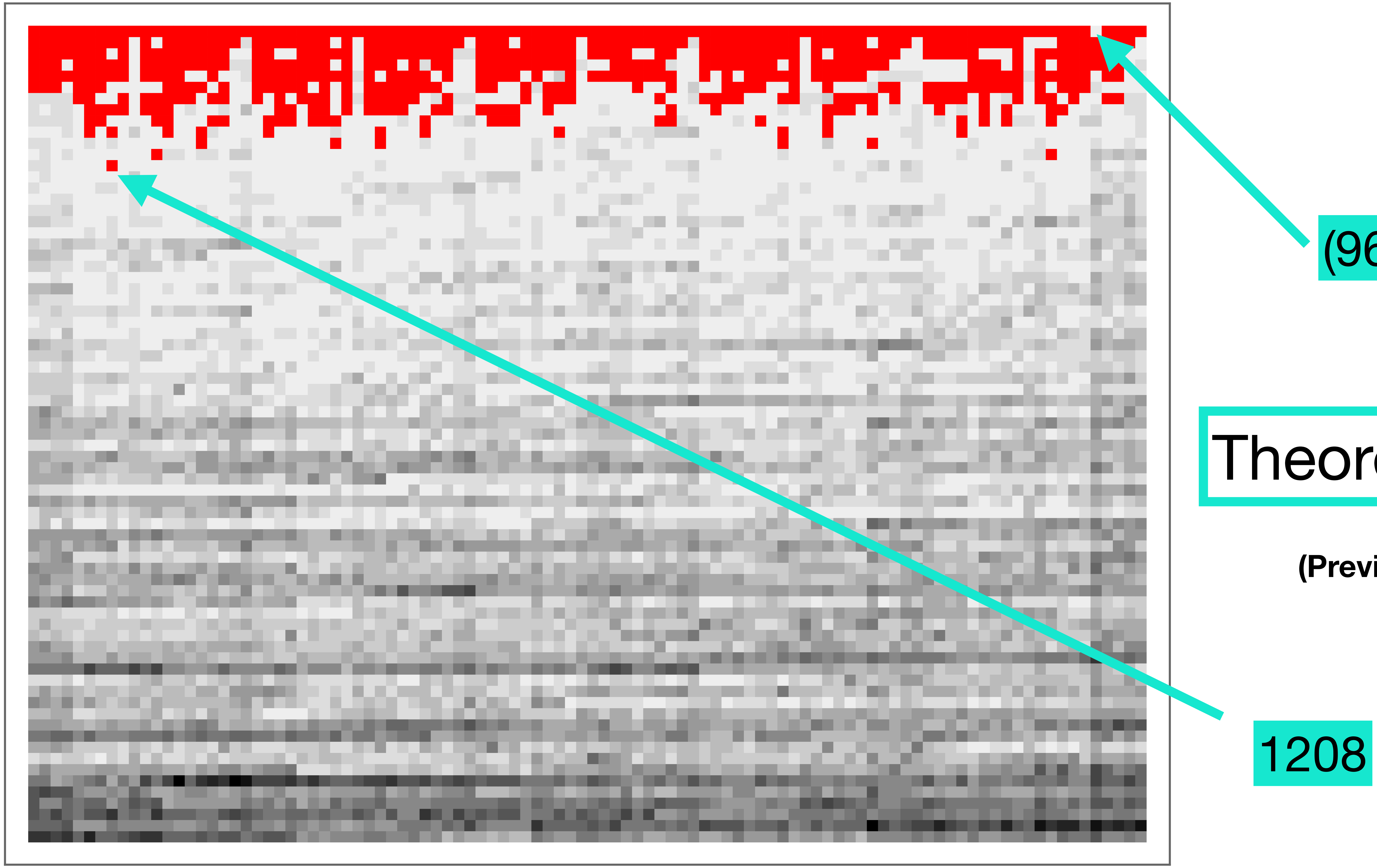


A-series (144_6)
[(112_6), (128_6) not intelligible]



Ad hoc (96_6)

What about N_6?



Theorem: $N_6 \leq 1208$

(Previous bound was 7350)

Further directions…?

Thank you!