Cartan subalgebras in classifiable C*-algebras

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Outline

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- Questions

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- Our construction is based on joint work with Barlak. It produces C*-diagonals in all classifiable stably finite C*-algebras. It also produces C*-diagonals in certain non-simple AX-algebras and in Villadsen algebras (j.w. Raad).

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A careful analysis of our construction leads to the following:

Theorem (L): Every classifiable unital stably finite C*-algebra has a C*-diagonal with connected spectrum.

Theorem (L): Every classifiable stably finite unital C*-algebra with torsion-free K_0 and trivial K_1 has a C*-diagonal whose spectrum is homeomorphic to the Menger curve.

The Menger curve

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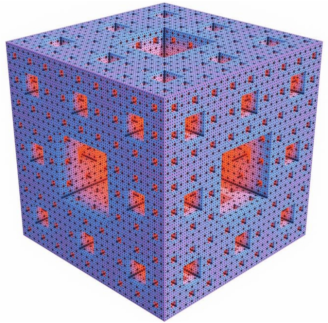

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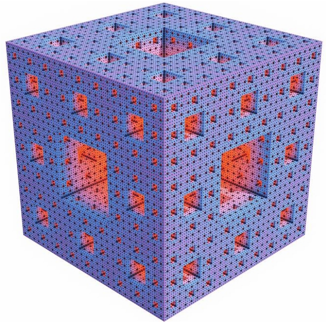

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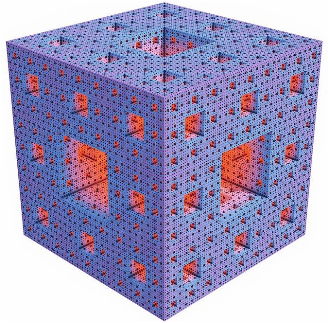

- The Menger curve is the universal 1-dimensional space:
  Every separable metrizable space of dimension ≤ 1 embeds into it.
- The Menger curve is the unique one-dimensional Peano continuum with no local cut points and no non-empty planar open subsets (Anderson)

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with torsion-free K_0 and trivial K_1
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- A similar result also holds in the stably projectionless setting.
  The Menger curve $\boldsymbol{M}$ has to be replaced by a locally compact but non-compact analogue of the form $\boldsymbol{M} \setminus C$,
  where C is a non-locally-separating copy of the Cantor space in $\boldsymbol{M}$.

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Open question

Does there exist a minimal homeomorphism on the Menger curve?

End

Thank you very much!