July 13, 2021

SHIFTING ANY PATH TO AN AVOIDABLE ONE

Matjaž Krnc
University of Primorska

joint work with V. Gurvich,
M. Milanič and M. Vyalyi

8th European Congress of Mathematics

Algorithmic Graph Theory (MS - ID 54)

OUTLINE

Gurvich, K., Milanič, Vyalyi(2021),
In an arbitrary graph, any induced path can be shifted to an avoidable one.

Beisegel, Chudnovsky, Gurvich, Milanič, Servatius (2019),
Bonamy, Defrain, Hatzel, Thiebaut (2020):
Every graph that has an induced P_k also has an **avoidable induced** P_k.

Case $k = 1$

In a $C_{\geq k+3}$-free graph, a P_k is avoidable iff it is simplicial

Ohtsuki-Cheung-Fujisawa (1976)
Every graph has an **avoidable vertex**.

Chvátal-Rusu-Sritharan (2002):
Every $\{C_{k+3}, C_{k+4} \ldots\}$-free graph that has an induced P_k also has a **simplicial induced** P_k.

In a chordal graph, a vertex is avoidable iff it is simplicial

Case $k = 1$

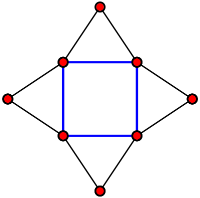
Dirac (1961):
Every chordal graph has a **simplicial vertex**.

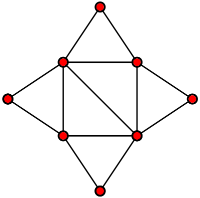
CHORDAL GRAPHS

A graph G is chordal if every cycle in G of length at least four has a chord.

CHORDAL GRAPHS

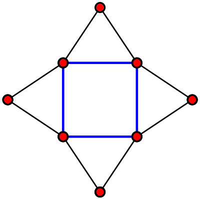
A graph G is chordal if every cycle in G of length at least four has a chord.

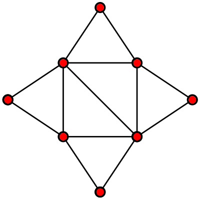




CHORDAL GRAPHS

A graph G is chordal if every cycle in G of length at least four has a chord.





Chordal graphs are known to have several structural and algorithmic properties.

CHORDAL GRAPHS

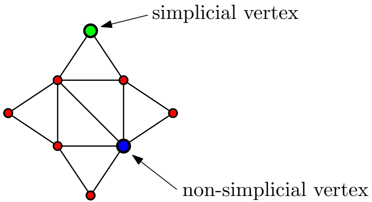
Theorem (Dirac 1961)

Every chordal graph has a simplicial vertex, that is, a vertex whose neighborhood is a clique.

CHORDAL GRAPHS

Theorem (Dirac 1961)

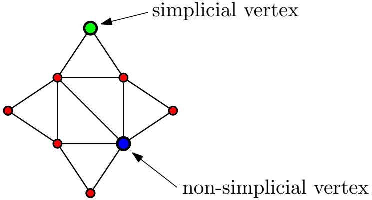
Every chordal graph has a simplicial vertex, that is, a vertex whose neighborhood is a clique.



CHORDAL GRAPHS

Theorem (Dirac 1961)

Every chordal graph has a simplicial vertex, that is, a vertex whose neighborhood is a clique.

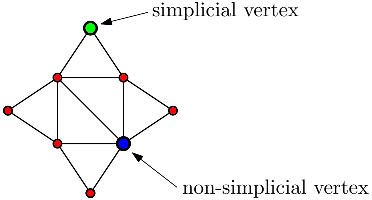


Simplicial vertices provide another characterization of chordal graphs, and were generalized in the literature in various ways:

CHORDAL GRAPHS

Theorem (Dirac 1961)

Every chordal graph has a simplicial vertex, that is, a vertex whose neighborhood is a clique.



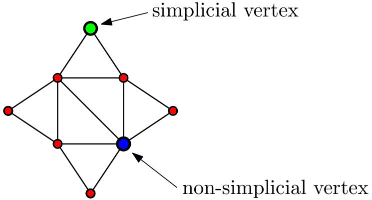
Simplicial vertices provide another characterization of chordal graphs, and were generalized in the literature in various ways:

- by generalizing simpliciality to a concept in general graphs;

CHORDAL GRAPHS

Theorem (Dirac 1961)

Every chordal graph has a simplicial vertex, that is, a vertex whose neighborhood is a clique.



Simplicial vertices provide another characterization of chordal graphs, and were generalized in the literature in various ways:

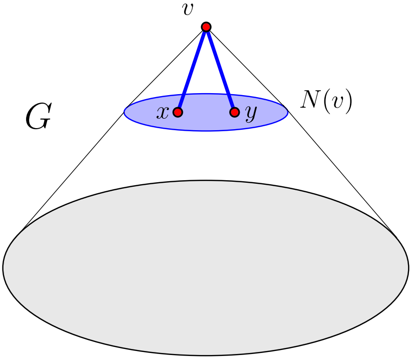
- by generalizing simpliciality to a concept in general graphs;
- by generalizing the 'simpliciality' property from vertices, which are paths of length 0, to longer induced paths.

First generalization:
from chordal graphs to all graphs

AVOIDABLE VERTICES

Definition (Ohtsuki et al. (1976))

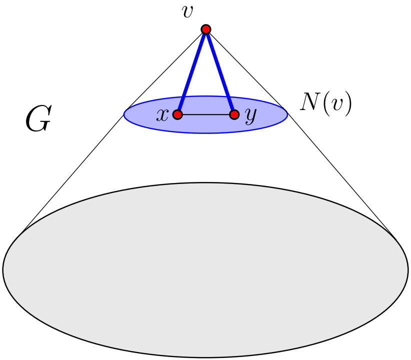
A vertex v in a graph G is avoidable if between any pair x and y of neighbors of v there exists an x, y-path all the internal vertices of which avoid v and all neighbors of v.



AVOIDABLE VERTICES

Definition (Ohtsuki et al. (1976))

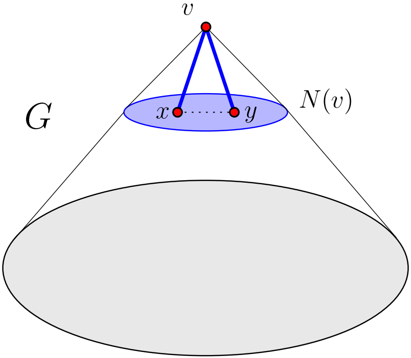
A vertex v in a graph G is avoidable if between any pair x and y of neighbors of v there exists an x, y-path all the internal vertices of which avoid v and all neighbors of v.



AVOIDABLE VERTICES

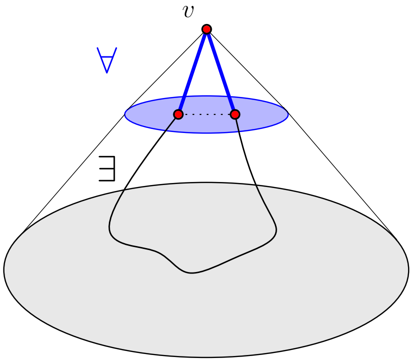
Definition (Ohtsuki et al. (1976))

A vertex v in a graph G is avoidable if between any pair x and y of neighbors of v there exists an x, y-path all the internal vertices of which avoid v and all neighbors of v.



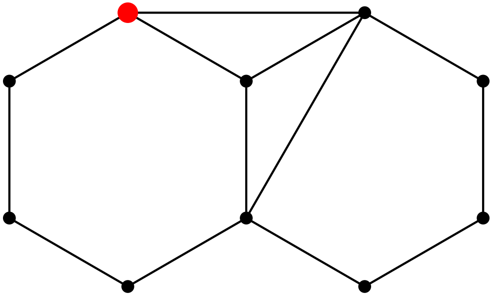
AVOIDABLE VERTICES

A vertex v is avoidable if and only if every induced P_3 with midpoint v closes to an induced cycle.



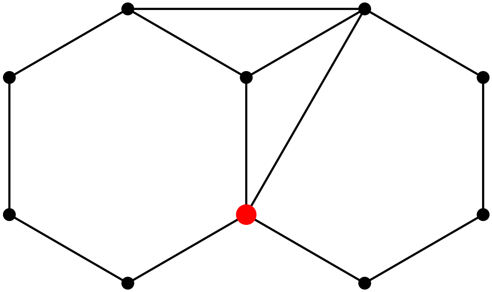
AVOIDABLE VERTICES

This vertex is avoidable:



AVOIDABLE VERTICES

However, this other one is not:



AVOIDABLE VERTICES

In a tree ($\neq K_1$) a vertex is avoidable if and only if it is a leaf.

AVOIDABLE VERTICES

In a tree ($\neq K_1$) a vertex is avoidable if and only if it is a leaf.

In a chordal graph a vertex is avoidable if and only if it is simplicial.

AVOIDABLE VERTICES

In a tree ($\neq K_1$) a vertex is avoidable if and only if it is a leaf.

In a chordal graph a vertex is avoidable if and only if it is simplicial.

Theorem (Ohtsuki, Cheung, Fujisawa, 1976)

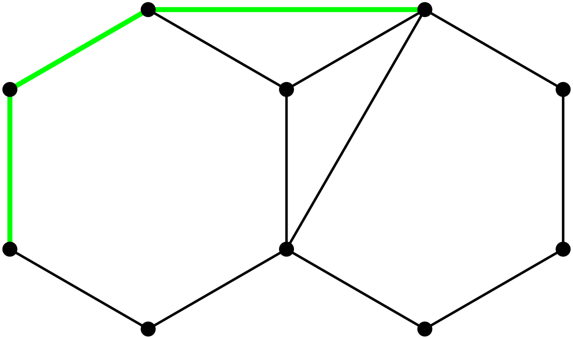
Every graph has an avoidable vertex.

Second generalization: from vertices to longer paths

SIMPLICIAL PATHS

Definition

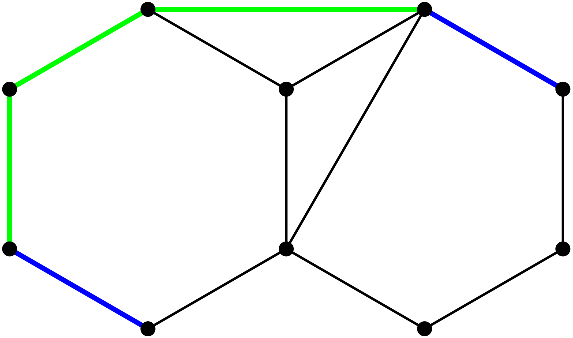
Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.



SIMPLICIAL PATHS

Definition

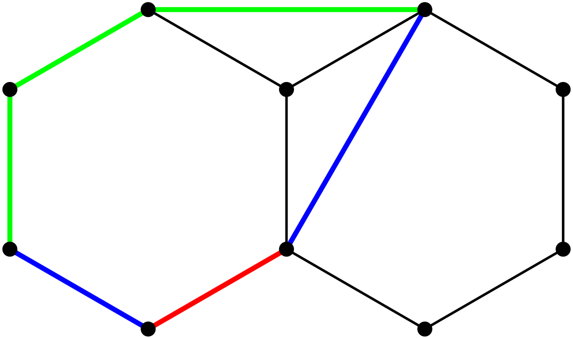
Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.



SIMPLICIAL PATHS

Definition

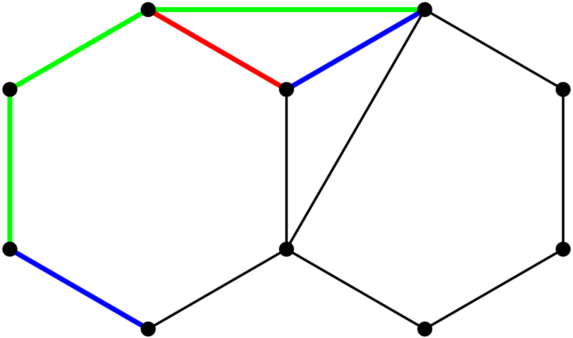
Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.



SIMPLICIAL PATHS

Definition

Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.

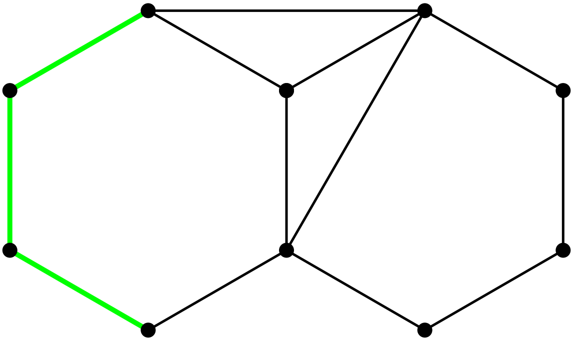


SIMPLICIAL PATHS

Definition

Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.

This green path is simplicial:

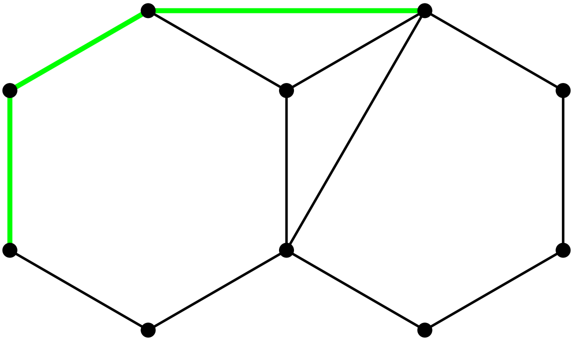


SIMPLICIAL PATHS

Definition

Given an induced path P in a graph G, an extension of P is any induced path in G obtained by adding to P one edge at each end.

However, this other green path is not simplicial:z



SIMPLICIAL PATHS

In this terminology, the result on chordal graphs can be formulated as follows.

SIMPLICIAL PATHS

In this terminology, the result on chordal graphs can be formulated as follows.

Every graph without induced cycles of length more than 3 has a simplicial induced P_1.

SIMPLICIAL PATHS

In this terminology, the result on chordal graphs can be formulated as follows.

Every graph without induced cycles of length more than 3 has a simplicial induced P_1.

Theorem (Chvátal, Rusu, Sritharan, 2002)

For each $k \geq 1$, every graph without induced cycles of length more than $k+2$ that has an induced P_k, has a simplicial induced P_k.

GENERALIZATIONS OF SIMPLICIAL VERTICES

Ohtsuki-Cheung-Fujisawa (1976)

Every graph has an **avoidable vertex**.

Chvátal-Rusu-Sritharan (2002):

Every $\{C_{k+3}, C_{k+4} \ldots\}$-free graph
that has an induced P_k
also has a **simplicial induced** P_k.

In a chordal graph,
a vertex is avoidable iff it is simplicial

Case $k = 1$

Dirac (1961):

Every chordal graph has a **simplicial vertex**.

A common generalization: avoidable paths!

AVOIDABLE PATHS

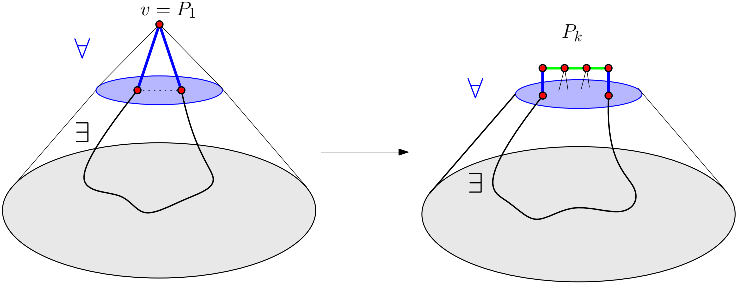
Definition

An induced path P in a graph G is avoidable if every extension of P is contained in an induced cycle.

AVOIDABLE PATHS

Definition

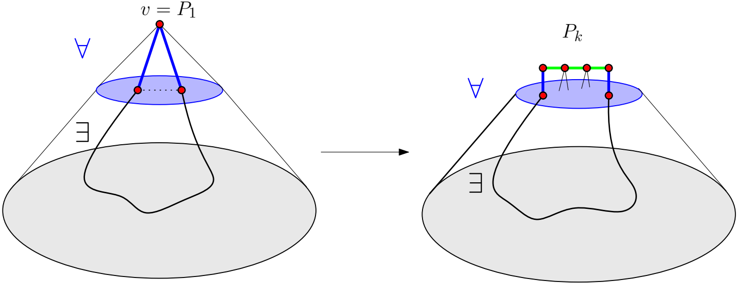
An induced path P in a graph G is avoidable if every extension of P is contained in an induced cycle.



AVOIDABLE PATHS

Definition

An induced path P in a graph G is avoidable if every extension of P is contained in an induced cycle.

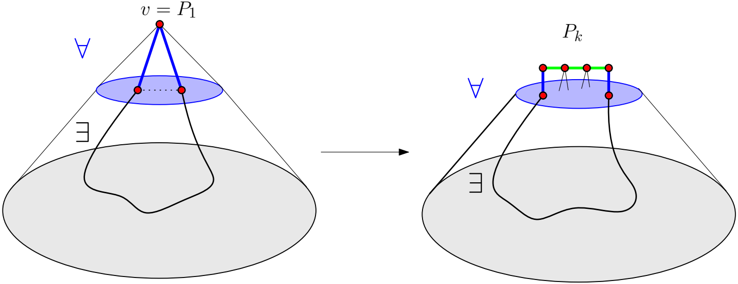


avoidable vertex $v \equiv$ avoidable P_1

AVOIDABLE PATHS

Definition

An induced path P in a graph G is avoidable if every extension of P is contained in an induced cycle.

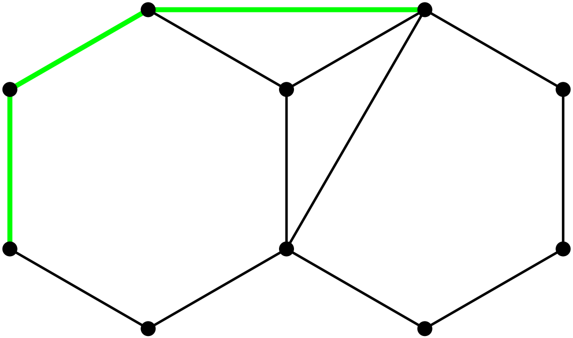


avoidable vertex $v \equiv$ avoidable P_1

Every simplicial induced path is (trivially) avoidable.

AVOIDABLE PATHS

This green path is not avoidable:



AVOIDABLE PATHS

Conjecture (Beisegel, Chudnovsky, Gurvich, Milanič, Servatius)

For every $k \geq 1$, every graph that has an induced P_k also has an avoidable induced P_k.

Theorem (Beisegel, Chudnovsky, Gurvich, Milanič, Servatius)

Every graph that has an edge also has an avoidable edge.

AVOIDABLE PATHS

Theorem (Beisegel, Chudnovsky, Gurvich, Milanič, Servatius)

Every graph that has an edge also has an avoidable edge.

Theorem (Bonamy, Defrain, Hatzel, Thiebaut (2020))

For every $k \geq 1$, every graph that has an induced P_k also has an avoidable induced P_k.

COMMON GENERALIZATION

Beisegel, Chudnovsky, Gurvich, Milanič, Servatius (2019),

Bonamy, Defrain, Hatzel, Thiebaut (2020):

Every graph that has an induced P_k
also has an **avoidable induced** P_k.

Case $k = 1$

In a $C_{\geq k+3}$-free graph,
a P_k is avoidable iff it is simplicial

Ohtsuki-Cheung-Fujisawa (1976)

Every graph has an **avoidable vertex**.

Chvátal-Rusu-Sritharan (2002):

Every $\{C_{k+3}, C_{k+4} \ldots\}$-free graph
that has an induced P_k
also has a **simplicial induced** P_k.

In a chordal graph,
a vertex is avoidable iff it is simplicial

Case $k = 1$

Dirac (1961):

Every chordal graph has a **simplicial vertex**.

Shifting any path to an avoidable one

AVOIDABLE PATHS AND SHIFTING

Let P and P' be induced k-paths in G.

We say that an P' is a shift of P if the vertices of $P \cup P'$ induce a $(k+1)$-path in G.

AVOIDABLE PATHS AND SHIFTING

Let P and P' be induced k-paths in G.

- We say that an P' is a shift of P if the vertices of $P \cup P'$ induce a $(k+1)$-path in G.
- We say that P is shiftable to P' and write $P \underset{G}{\leftrightharpoons} P'$ if it can be obtained from P' by a sequence of shifts.

AVOIDABLE PATHS AND SHIFTING

Let P and P' be induced k-paths in G.

We say that an P' is a shift of P if the vertices of $P \cup P'$ induce a $(k+1)$-path in G.

We say that P is shiftable to P' and write $P \underset{G}{\leftrightharpoons} P'$ if it can be obtained from P' by a sequence of shifts.

Theorem (Gurvich, K., Milanič, Vyalyi)

In every graph, every induced path can be shifted to an avoidable one.

AVOIDABLE PATHS AND SHIFTING

For a graph G and a positive integer k,

- property $H_B(G, k)$ holds if every induced P_k in G can be shifted to an avoidable one;

AVOIDABLE PATHS AND SHIFTING

For a graph G and a positive integer k,

- property $H_B(G, k)$ holds if every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G, k, v)$ (where $v \in V(G)$) holds if every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;

AVOIDABLE PATHS AND SHIFTING

For a graph G and a positive integer k,

- property $H_B(G, k)$ holds if every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G, k, v)$ (where $v \in V(G)$) holds if every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;
- property $H_R(G, k)$ holds if $H_R(G, k, v)$, for every $v \in V(G)$.

AVOIDABLE PATHS AND SHIFTING

For a graph G and a positive integer k,

- property $H_B(G,k)$ holds if every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G,k,v)$ (where $v \in V(G)$) holds if every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;
- property $H_R(G,k)$ holds if $H_R(G,k,v)$, for every $v \in V(G)$.

Lemma

$H_R(G,k)$ *implies* $H_B(G,k)$.

AVOIDABLE PATHS AND SHIFTING

For a graph G and a positive integer k,

- property $H_B(G,k)$ holds if every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G,k,v)$ (where $v \in V(G)$) holds if every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;
- property $H_R(G,k)$ holds if $H_R(G,k,v)$, for every $v \in V(G)$.

Lemma

$H_R(G,k)$ *implies* $H_B(G,k)$.

So it is enough to prove:

Lemma

For any graph G and positive integer k, property $H_R(G,k)$ holds.

Lemma

For any graph G and positive integer k, property $H_R(G,k)$ holds.

Recall the ingredients:

- property $H_B(G,k)$ – every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G,k,v)$ – every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;
- property $H_R(G,k)$ – if $H_R(G,k,v)$, for every $v \in V(G)$.
- Lemma: $H_R(G,k)$ implies $H_B(G,k)$.

Lemma

For any graph G and positive integer k, property $H_R(G,k)$ holds.

Recall the ingredients:

- property $H_B(G,k)$ – every induced P_k in G can be shifted to an avoidable one;
- property $H_R(G,k,v)$ – every induced P_k in $G - N[v]$ can be shifted in $G - N[v]$ to an avoidable path in G;
- property $H_R(G,k)$ – if $H_R(G,k,v)$, for every $v \in V(G)$.
- Lemma: $H_R(G,k)$ implies $H_B(G,k)$.
- Lemma [BDHT (2020)]: Let $G' = G|_{u_1 u_2 \to u}$. If an induced k-path Q in $G' - N[u]$ is avoidable in G', it is avoidable in G as well.

.

Procedure 1 $\textsc{Shifting}(G, P)$

Input: a graph G and an induced path $P = p_1 p_2 \ldots p_k$ in G
Output: a sequence S of paths shifting P to an avoidable induced path in G

if there exists an extension xPy of P **then**
    $Q \leftarrow y p_k \ldots p_1 x$
    return $P, \textsc{RefinedShifting}(G, Q)$
else
    return P

Procedure 2 $\textsc{RefinedShifting}(G, P)$

Input: a graph G and an induced path $P = p_1 \ldots p_{k+2}$ in G
Output: a sequence S of paths in $G - N[p_{k+2}]$ shifting $p_1 \ldots p_k$ to an avoidable induced path in G

$P' \leftarrow p_1 \ldots p_k$
$S \leftarrow$ the one-element sequence containing path P'
if there exists an extension $xP'y$ in $G - N[p_{k+2}]$ **then**
    $S \leftarrow S, \textsc{RefinedShifting}(G - N[p_{k+2}], xP'y)$
$Q \leftarrow$ the end path of S
if Q has an extension xQy in G such that y is the unique neighbor of p_{k+2} in $\{x, y\}$ **then**
    let $Q = q_1 \ldots q_k$ such that y is adjacent to q_k
    $Q' \leftarrow x q_1 \ldots q_k$
    $G' \leftarrow G/_{p_{k+2} y \to y'}$
    $S' \leftarrow \textsc{RefinedShifting}(G', Q'y')$
    return S, S'
else
    return S

Gurvich, K., Milanič, Vyalyi(2021),
In an arbitrary graph, any induced path can be shifted to an avoidable one.

Beisegel, Chudnovsky, Gurvich, Milanič, Servatius (2019),
Bonamy, Defrain, Hatzel, Thiebaut (2020):
Every graph that has an induced P_k also has an **avoidable induced** P_k.

Case $k = 1$

In a $C_{\geq k+3}$-free graph, a P_k is avoidable iff it is simplicial

Ohtsuki-Cheung-Fujisawa (1976)
Every graph has an **avoidable vertex**.

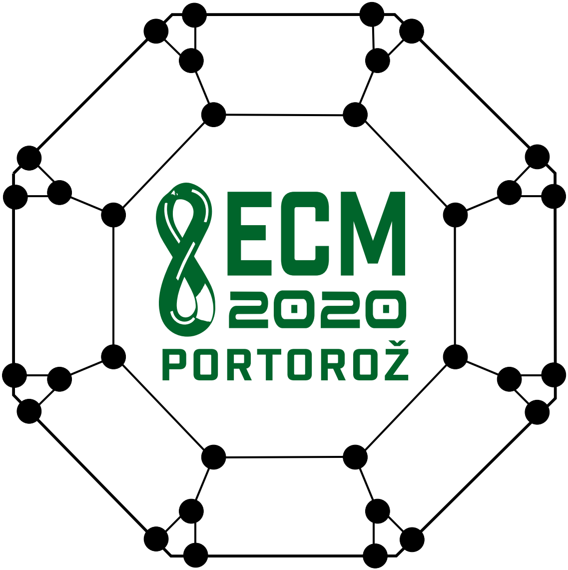
Chvátal-Rusu-Sritharan (2002):
Every $\{C_{k+3}, C_{k+4} \ldots\}$-free graph that has an induced P_k also has a **simplicial induced** P_k.

In a chordal graph, a vertex is avoidable iff it is simplicial

Case $k = 1$

Dirac (1961):
Every chordal graph has a **simplicial vertex**.

Thank you!



Find a shortest induced path which is not avoidable.