Limits for embedding distributions

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Genus polynomial

- By the genus distribution of a graph G, we mean the sequence

$$\gamma_0(G), \gamma_1(G), \gamma_2(G), \cdots,$$

where $\gamma_i(G)$ is the number of distinct embeddings of G with genus i for $i \geq 0$.

- The genus polynomial of G is $\Gamma_G(x) = \sum_{k=0}^{\infty} \gamma_k(G) x^k$.

Probability genus polynomial

- For any graph G, let X_G be a random variable with distribution

$$p_i = \mathbb{P}(X_G = i) = \frac{\gamma_i(G)}{\Gamma_G(1)}, \quad i = 0, 1, \cdots . \tag{1}$$

- The *probability genus polynomial* of G is defined as

$$P_{X_G}(z) = \sum_{i \geq 0} p_i z^i.$$

- The *probability genus distribution* of G is the sequence $p_0, p_1, \ldots, .$

Crosscap-number distribution and Euler-genus distribution

- The crosscap-number distribution : $\tilde{\gamma}_1(G), \tilde{\gamma}_2(G), \cdots$
- The crosscap-number polynomial: $\tilde{\Gamma}_G(x) = \sum_{j=1}^{\infty} \tilde{\gamma}_j(G)x^j$.
- The Euler-genus distribution: $\varepsilon_0(G), \varepsilon_1(G), \varepsilon_2(G), \cdots$
- The Euler-genus polynomial: $\varepsilon_G(x) = \sum_{i=0}^{\infty} \varepsilon_i(G)x^i$.
- Similarly, we have the *probability Euler-genus (crosscap-number) polynomial* of G

- For any graph G, it holds that

$$\epsilon_G(x) = \Gamma_G(x^2) + \tilde{\Gamma}_G(x). \quad (2)$$

- When we say embedding distribution of a graph G, we mean its genus distribution, crosscap-number distribution or Euler-genus distribution.

Global feature for embedding distribution

- Log-concavity, conjectured by Gross, Robbins, and Tucker (1989), Chen, and Gross (2018)

- Partial results obtained by Gross and his coauthors, Stahl (1997) et. al..

- Mean of embedding distribution or Average genus, average crosscap-number and average Euler-genus.

- Archdeacon(1989), J. Chen, Gross, and Rieper(1992,1996), Y. Chen and Y. Liu(2006), Stahl(1992,1996), White(1996) Y. Chen (2018++), Zhang, peng and Chen(2019), et. al..

- Variance. Stahl (1996).

Limit for probability embedding distribution X_n.

- Problem: when n is big enough, whether the distribution X_n will converge to some well-known distribution in probability.

- If the answer is yes, then it demonstrates the outline of embedding distribution for graph G_n when n is big enough.

Asymptotically normal distribution

- Suppose $\{G_n\}_{n=1}^{\infty}$ is a sequence of graphs. For $n \geq 1$, let e_n and v_n be the mean and variance of embedding distribution of G_n, respectively. We say the embedding distribution of G_n is asymptotically normal distribution when n tends to infinity if for any $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_{G_n} - e_n}{\sqrt{v_n}} \leq x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \mathrm{d}u.$$

Normal distribution

- Two interesting properties of normal distributions:
  (1) Symmetry, normal distributions are symmetric around their mean.
  (2) Approximately 95% of the area of a normal distribution is within two standard deviations of the mean.

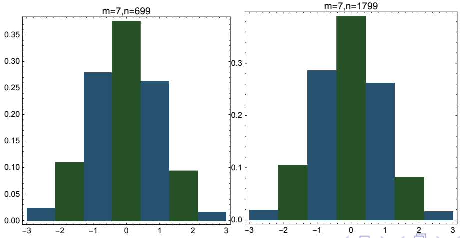
Asymptotically normal distribution

- If the embedding distribution of G_n is asymptotically normal distribution, then the number of embeddings of G_n are mainly concentrated on the interval $(e_n - 2\sqrt{v_n}, e_n + 2\sqrt{v_n})$ when n is big enough.

Numeric Simulations for ladder graph $L_n = K_2 \square P_n$

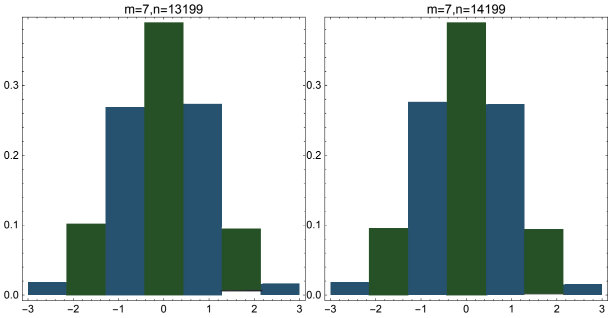
- The genus distribution of ladders was obtained by Furst, Gross and Stateman (1987).
- Let $c > 0$ be a constant. We divide the interval $(-c, c]$ into m intervals

$$I_i = (u_{i-1}, u_i] := (-c + 2(i-1)c/m, -c + 2ic/m],$$



Numeric Simulations for ladder graphs

- $m = 7, n = 13199$, and $m = 7, n = 14199$



Graph sequences with bounded maximum genus

- A sequence $\{G_n\}_{n=1}^{\infty}$ of graphs is called strictly monotone sequence if no pair of graphs in the sequence are homeomorphic and each G_i is homeomorphic to a subgraph of G_{i+1} for all $i > 1$.
- We say a strictly monotone graph sequence $\{G_n\}_{n=1}^{\infty}$ bounded if there exists a positive constant C such that $\gamma_{max}(G_i) \leq C$ for $i \geq 1$.

A central limit theorem for $\{G_n\}_{n=1}^{\infty}$

Theorem (B)

Let $\{G_n\}_{n=1}^{\infty}$ be a strictly monotone sequence of connected graphs. If $\{G_n\}_{n=1}^{\infty}$ is bounded, then the Euler-genus distribution of G_n is asymptotically normal distribution.

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Sketch of proof

- Sketch of proof. Since the strictly monotone sequence of connected graphs $G_1, G_2, G_3, \cdots$, is bounded, then the values of the maximum genus of the graphs approach a finite limit point, and there exists an index N such that all but a finite number of graphs in the sequence can be obtained by attaching ears serially or by bar-amalgamation of a cactus to G_N. The resulting graph is denoted $G_{r,s,t}$. Finally we prove that the Euler-genus distribution of $G_{r,s,t}$ is asymptotically normal distribution.

Bar-amalgamation

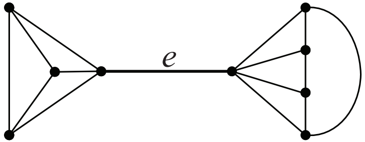
- A bar-amalgamation $G \oplus_e H$ of two disjoint graphs H and G is obtained by adding a new edge $e = uv$ between a vertex u of G and a vertex v of H.
- 

Figure: $K_4 \oplus_e (K_5 - 2K_2)$

Tree-like graphs families

- Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of connected graphs. A sequence of tree-like graphs $\{G_n\}_{n=1}^{\infty}$ are obtained in the following ways:
  - (A) For $n = 1$, $G_1 = H_1$.
  - (B) If we have obtained the graph G_{n-1}, the graph G_n is obtained by adding an edge between a vertex of G_{n-1} and a vertex of H_n. I.e, $G_n = G_{n-1} \oplus_e H_n$.

A Proposition

- Proposition 1 Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent random variables with finite second moments. Let $\sigma_k^2 = E\xi_k^2 - (E\xi_k)^2 > 0, B_n = \sqrt{\sum_{k=1}^{n} \sigma_k^2}$. Assume that for a sequence of positive constants $\{C_n\}_{n=1}^{\infty}$, we have $\sup_{1\le k\le n} |\xi_k| \le C_n$, and $\lim_{n\to\infty} \frac{C_n}{B_n} = 0$. Then, it holds that

$$\sup_{x\in R} \left| \mathbb{P}\left(\frac{\sum_{k=1}^{n}(\xi_k - E\xi_k)}{B_n} \le x\right) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t \right| = 0$$

when n tends to infinity.

Theorem (A)

Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of tree-like graphs. Assume the followings holds.

- *For each n, H_n is a finite connected graph.*
- *Except a finite number of $n \in N$, $\gamma_{\max}(H_n) > \gamma_{\min}(H_n)$.*

Then, the genus distribution of G_n is asymptotically normal distribution with mean e_n and variance v_n.

Sketch of proof

- Sketch of proof. Step 1. In this step, we prove that: for two positive constants c, C, it holds that

$$0 < c \le \inf_{n \in N} \Gamma_{var}(H_n) \le \sup_{n \in N} \Gamma_{var}(H_n) \le C \tag{3}$$

and

$$0 < c \le \inf_{n \in N} \Gamma_{avg}(H_n) \le \sup_{n \in N} \Gamma_{avg}(H_n) \le C. \tag{4}$$

- Step 2. For each $n \in N$, the distribution of random variable $\xi_1 + \cdots + \xi_n$ is given by

$$\mathbb{P}(\xi_1 + \cdots + \xi_n = j) = \frac{\gamma_j(G_n)}{\Gamma_{G_n}(1)}, j = 0, 1, \cdots. \quad (5)$$

- Step 3. In this step, we give a proof of our theorem using Proposition 1.

Thank you!