Computing distance-regular graph and association scheme parameters in SageMath with `sage-drg`

Janoš Vidali, University of Ljubljana

Based on joint work with Alexander Gavrilyuk, Aleksandar Jurišić,
Sho Suda and Jason Williford

Live slides on Binder

`https://github.com/jaanos/sage-drg`

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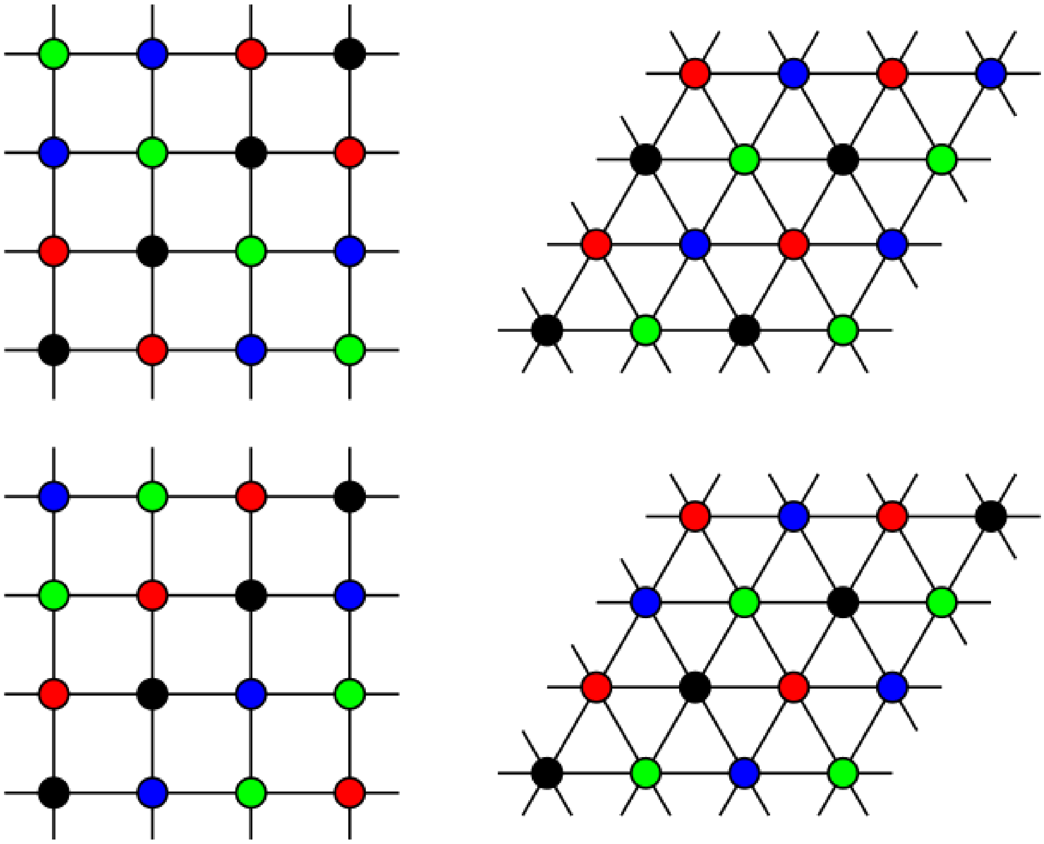
Association schemes

- Association schemes were defined by Bose and Shimamoto in 1952 as a theory underlying experimental design.
- They provide a unified approach to many topics, such as
  - combinatorial designs,
  - coding theory,
  - generalizing groups, and
  - strongly regular and distance-regular graphs.

Examples

- Hamming schemes: $X = \mathbb{Z}_n^d$, $x\ R_i\ y \Leftrightarrow \text{weight}(x - y) = i$
- Johnson schemes: $X = \{S \subseteq \mathbb{Z}_n \mid |S| = d\}$ $(2d \leq n)$, $x\ R_i\ y \Leftrightarrow |x \cap y| = d - i$



Definition

- Let X be a set of vertices and $\mathcal{R} = \{R_0 = \text{id}_X, R_1, \ldots, R_D\}$ a set of symmetric relations partitioning X^2.
- $(X, \mathcal{R})$ is said to be a D-class association scheme if there exist numbers p_{ij}^h $(0 \leq h, i, j \leq D)$ such that, for any $x, y \in X$,

$$x\ R_h\ y \Rightarrow |\{z \in X \mid x\ R_i\ z\ R_j\ y\}| = p_{ij}^h$$

- We call the numbers p_{ij}^h $(0 \leq h, i, j \leq D)$ intersection numbers.

Main problem

- Does an association scheme with given parameters exist?
  - If so, is it unique?
  - Can we determine all such schemes?
- Lists of feasible parameter sets have been compiled for strongly regular and distance-regular graphs.
- Recently, lists have also been compiled for some Q-polynomial association schemes.
- Computer software allows us to efficiently compute parameters and check for existence conditions, and also to obtain new information which would be helpful in the construction of new examples.

Bose-Mesner algebra

- Let A_i be the binary matrix corresponding to the relation R_i $(0 \le i \le D)$.
- The vector space $\mathcal{M}$ over $\mathbb{R}$ spanned by A_i $(0 \le i \le D)$ is called the Bose-Mesner algebra.
- $\mathcal{M}$ has a second basis $\{E_0, E_1, \dots, E_D\}$ consisting of projectors to the common eigenspaces of A_i $(0 \le i \le D)$.
- There exist the eigenmatrix P and the dual eigenmatrix Q such that

$$A_j = \sum_{i=0}^{D} P_{ij} E_i, \qquad E_j = \frac{1}{|X|} \sum_{i=0}^{D} Q_{ij} A_i.$$

- There are nonnegative constants q^h_{ij}, called Krein parameters, such that

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{D} q^h_{ij} E_h,$$

  where $\circ$ is the entrywise matrix product.

Parameter computation: general association schemes

[2]:
```
%display latex
import drg
p = [[[1, 0, 0, 0], [0, 6, 0, 0], [0, 0, 3, 0], [0, 0, 0, 6]],
     [[0, 1, 0, 0], [1, 2, 1, 2], [0, 1, 0, 2], [0, 2, 2, 2]],
     [[0, 0, 1, 0], [0, 2, 0, 4], [1, 0, 2, 0], [0, 4, 0, 2]],
     [[0, 0, 0, 1], [0, 2, 2, 2], [0, 2, 0, 1], [1, 2, 1, 2]]]
scheme = drg.ASParameters(p)
scheme.kreinParameters()
```

[2]:

$$0 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$1 : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

$$2 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$$

$$3 : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Metric and cometric schemes

- If $p^h_{ij} \neq 0$ (resp. $q^h_{ij} \neq 0$) implies $|i-j| \leq h \leq i+j$, then the association scheme is said to be metric (resp. cometric).
- The parameters of a metric (or P-polynomial) association scheme can be determined from the intersection array

$$\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\} \quad (b_i = p^i_{1,i+1}, c_i = p^i_{1,i-1}).$$

- The parameters of a cometric (or Q-polynomial) association scheme can be determined from the Krein array

$$\{b^*_0, b^*_1, \ldots, b^*_{D-1}; c^*_1, c^*_2, \ldots, c^*_D\} \quad (b^*_i = q^i_{1,i+1}, c^*_i = q^i_{1,i-1}).$$

- Metric association schemes correspond to distance-regular graphs.

Parameter computation: metric and cometric schemes

[3]:
```
from drg import DRGParameters
syl = DRGParameters([5, 4, 2], [1, 1, 4])
syl
```

[3]: Parameters of a distance-regular graph with intersection array $\{5, 4, 2; 1, 1, 4\}$

[4]:
```
syl.order()
```

[4]: 36

[5]:
```
from drg import QPolyParameters
q225 = QPolyParameters([24, 20, 36/11], [1, 30/11, 24])
q225
```

[5]: Parameters of a Q-polynomial association scheme with Krein array $\{24, 20, \frac{36}{11}; 1, \frac{30}{11}, 24\}$

[6]:
```
q225.order()
```

[6]: 225

[7]:
```
syl.pTable()
```

[7]:

$$0: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$1: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 4 & 8 & 8 \\ 0 & 0 & 8 & 2 \end{pmatrix}$$

$$2: \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 11 & 6 \\ 0 & 2 & 6 & 2 \end{pmatrix}$$

$$3: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 12 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}$$

[8]:

```
syl.kreinParameters()
```

[8]:

$$0: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$1: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \frac{44}{5} & \frac{22}{5} & \frac{9}{5} \\ 0 & \frac{22}{5} & 2 & \frac{18}{5} \\ 0 & \frac{9}{5} & \frac{18}{5} & \frac{18}{5} \end{pmatrix}$$

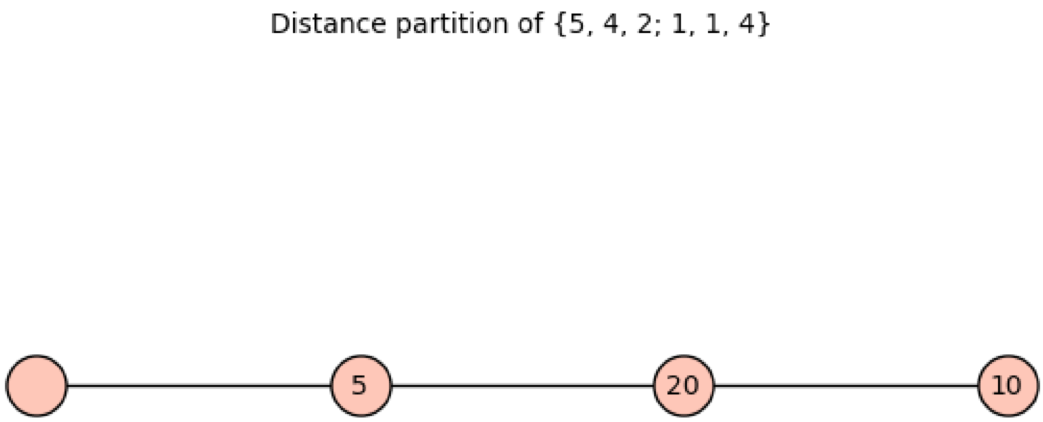
$$2: \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{176}{25} & \frac{16}{5} & \frac{144}{25} \\ 1 & \frac{16}{5} & 4 & \frac{9}{5} \\ 0 & \frac{144}{25} & \frac{9}{5} & \frac{36}{25} \end{pmatrix}$$

$$3: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{16}{5} & \frac{32}{5} & \frac{32}{5} \\ 0 & \frac{32}{5} & 2 & \frac{8}{5} \\ 1 & \frac{32}{5} & \frac{8}{5} & 0 \end{pmatrix}$$

[9]:

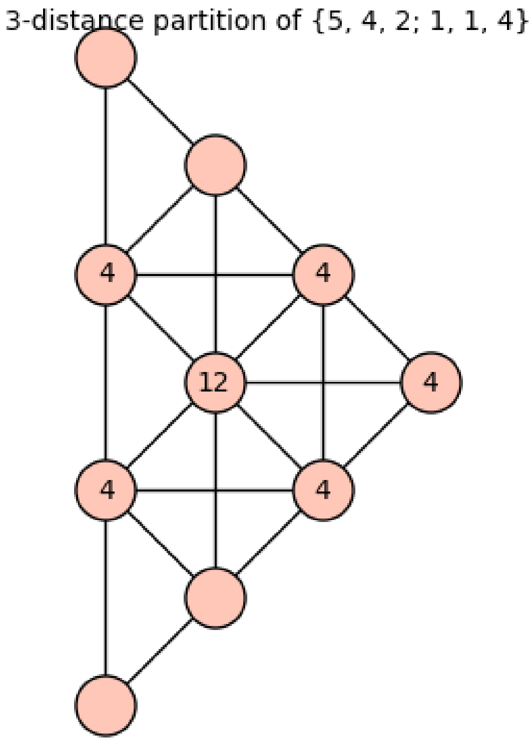
```
syl.distancePartition()
```

[9]:



[11]:
```
syl.distancePartition(3)
```

[11]:



Parameter computation: parameters with variables

Let us define a one-parametric family of intersection arrays.

[12]:
```
r = var("r")
f = DRGParameters([2*r^2*(2*r+1), (2*r-1)*(2*r^2+r+1), 2*r^2], [1, 2*r^2,␣
↪r*(4*r^2-1)])
f.order(factor=True)
```

[12]: $(2r+1)^3 r$

[13]:
```
f1 = f.subs(r == 1)
f1
```

[13]: Parameters of a distance-regular graph with intersection array $\{6,4,2;1,2,3\}$

The parameters of f1 are known to uniquely determine the Hamming scheme $H(3,3)$.

[14]:
```
f2 = f.subs(r == 2)
f2
```

[14]: Parameters of a distance-regular graph with intersection array $\{40,33,8;1,8,30\}$

Feasibility checking

A parameter set is called feasible if it passes all known existence conditions.

Let us verify that $H(3,3)$ is feasible.

[15]:
```
f1.check_feasible()
```

No error has occured, since all existence conditions are met.

Let us now check whether the second member of the family is feasible.

[16]:
```
f2.check_feasible()
```

```
...
InfeasibleError: nonexistence by JurišićVidali12
```

In this case, nonexistence has been shown by matching the parameters against a list of nonexistent families.

Triple intersection numbers

- In some cases, triple intersection numbers can be computed.

$$[h\ i\ j] = \begin{bmatrix} x & y & z \\ h & i & j \end{bmatrix} = |\{w \in X \mid w\ R_i\ x \wedge w\ R_j\ y \wedge w\ R_h\ z\}|$$

- If $x\ R_W\ y$, $x\ R_V\ z$ and $y\ R_U\ z$, then we have

$$\sum_{\ell=1}^{D} [\ell\ j\ h] = p^U_{jh} - [0\ j\ h], \qquad \sum_{\ell=1}^{D} [i\ \ell\ h] = p^V_{ih} - [i\ 0\ h], \qquad \sum_{\ell=1}^{D} [i\ j\ \ell] = p^W_{ij} - [i\ j\ 0],$$

  where

$$[0\ j\ h] = \delta_{jW}\delta_{hV}, \qquad [i\ 0\ h] = \delta_{iW}\delta_{hU}, \qquad [i\ j\ 0] = \delta_{iV}\delta_{jU}.$$

- Additionally, $q^h_{ij} = 0$ if and only if

$$\sum_{r,s,t=0}^{D} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} x & y & z \\ r & s & t \end{bmatrix} = 0$$

  for all $x, y, z \in X$.

Example: parameters for a bipartite DRG of diameter 5

We will show that a distance-regular graph with intersection array $\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}$ does not exist. The existence of such a graph would give a counterexample to a conjecture by MacLean and Terwilliger, see Bipartite distance-regular graphs: The Q-polynomial property and pseudo primitive idempotents by M. Lang.

[17]:
```
p = drg.DRGParameters([55, 54, 50, 35, 10], [1, 5, 20, 45, 55])
p.check_feasible(skip=["sporadic"])
p.order()
```

[17]:

3500

[18]:
```
p.kreinParameters()
```

[18]:

$$0: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 132 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1617 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1617 & 0 & 0 \\ 0 & 0 & 0 & 0 & 132 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$1: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{50}{3} & \frac{343}{3} & 0 & 0 & 0 \\ 0 & \frac{343}{3} & \frac{2450}{3} & 686 & 0 & 0 \\ 0 & 0 & 686 & \frac{2450}{3} & \frac{343}{3} & 0 \\ 0 & 0 & 0 & \frac{343}{3} & \frac{50}{3} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$2: \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{28}{3} & \frac{200}{3} & 56 & 0 & 0 \\ 1 & \frac{200}{3} & \frac{2380}{3} & 700 & 56 & 0 \\ 0 & 56 & 700 & \frac{2380}{3} & \frac{200}{3} & 1 \\ 0 & 0 & 56 & \frac{200}{3} & \frac{28}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$3: \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 56 & \frac{200}{3} & \frac{28}{3} & 0 \\ 0 & 56 & 700 & \frac{2380}{3} & \frac{200}{3} & 1 \\ 1 & \frac{200}{3} & \frac{2380}{3} & 700 & 56 & 0 \\ 0 & \frac{28}{3} & \frac{200}{3} & 56 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

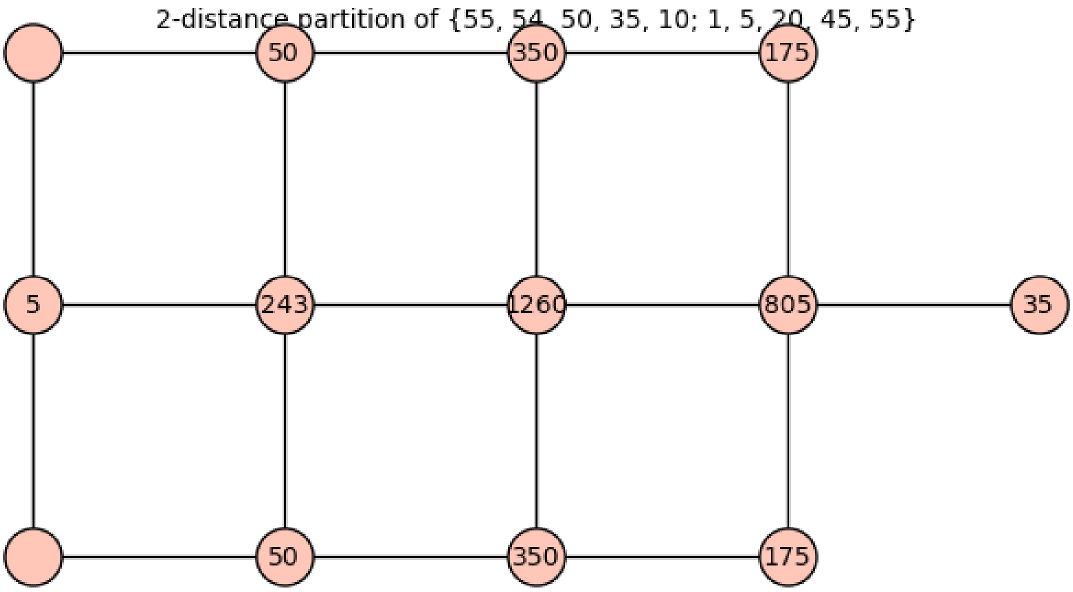
$$4: \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{343}{3} & \frac{50}{3} & 1 \\ 0 & 0 & 686 & \frac{2450}{3} & \frac{343}{3} & 0 \\ 0 & \frac{343}{3} & \frac{2450}{3} & 686 & 0 & 0 \\ 1 & \frac{50}{3} & \frac{343}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$5: \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 132 & 0 \\ 0 & 0 & 0 & 1617 & 0 & 0 \\ 0 & 0 & 1617 & 0 & 0 & 0 \\ 0 & 132 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We now compute the triple intersection numbers with respect to three vertices x, y, z at mutual distances 2. Note that we have $p^2_{22} = 243$, so such triples must exist. The parameter α will denote the number of vertices adjacent to all of x, y, z.

[19]:
```
p.distancePartition(2)
```

[19]:



[20]:
```
S222 = p.tripleEquations(2, 2, 2, params={"alpha": (1, 1, 1)})
show(S222[1, 1, 1])
show(S222[5, 5, 5])
```

α

$-12\,\alpha + 20$

Let us consider the set A of common neighbours of x and y, and the set B of vertices at distance 2 from both x and y. By the above, each vertex in B has at most one neighbour in A, so there are at most 243 edges between A and B. However, each vertex in A is adjacent to both x and y, and the other 53 neighbours are in B, amounting to a total of $5 \cdot 53 = 265$ edges. We have arrived to a contradiction, and we must conclude that a graph with intersection array $\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}$ does not exist.

Double counting

- Let $x, y \in X$ with $x\ R_r\ y$.
- Let $\alpha_1, \alpha_2, \dots \alpha_u$ and $\kappa_1, \kappa_2, \dots \kappa_u$ be numbers such that there are precisely κ_ℓ vertices $z \in X$ with $x\ R_s\ z\ R_t\ y$ such that

$$\begin{bmatrix} x & y & z \\ h & i & j \end{bmatrix} = \alpha_\ell \qquad (1 \le \ell \le u).$$

- Let $\beta_1, \beta_2, \dots \beta_v$ and $\lambda_1, \lambda_2, \dots \lambda_v$ be numbers such that there are precisely λ_ℓ vertices $w \in X$ with $x\ R_h\ w\ R_i\ y$ such that

$$\begin{bmatrix} w & x & y \\ j & s & t \end{bmatrix} = \beta_\ell \qquad (1 \le \ell \le v).$$

- Double-counting pairs (w, z) with $w\ R_j\ z$ gives

$$\sum_{\ell=1}^{u} \kappa_\ell \alpha_\ell = \sum_{\ell=1}^{v} \lambda_\ell \beta_\ell$$

- Special case: $u = 1, \alpha_1 = 0$ implies $v = 1, \beta_1 = 0$.

Example: parameters for a 3-class Q-polynomial scheme

Nonexistence of some Q-polynomial association schemes has been proven by obtaining a contradiction in double counting with triple intersection numbers.

[21]:
```
q225
```

[21]: Parameters of a Q-polynomial association scheme with Krein array $\{24, 20, \frac{36}{11}; 1, \frac{30}{11}, 24\}$

[22]:
```
q225.check_quadruples()
```

```
...
InfeasibleError: found forbidden quadruple wxyz with d(w, x) = 2, d(w, y) = 2,␣
↪d(w, z) = 2, d(x, y) = 3, d(x, z) = 3, d(y, z) = 3
```

Integer linear programming has been used to find solutions to multiple systems of linear Diophantine equations, eliminating inconsistent solutions.

More results

There is no distance-regular graph with intersection array

- $\{83, 54, 21; 1, 6, 63\}$ (1080 vertices)
- $\{135, 128, 16; 1, 16, 120\}$ (1360 vertices)
- $\{104, 70, 25; 1, 7, 80\}$ (1470 vertices)
- $\{234, 165, 12; 1, 30, 198\}$ (1600 vertices)
- $\{195, 160, 28; 1, 20, 168\}$ (2016 vertices)
- $\{125, 108, 24; 1, 9, 75\}$ (2106 vertices)
- $\{126, 90, 10; 1, 6, 105\}$ (2197 vertices)
- $\{203, 160, 34; 1, 16, 170\}$ (2640 vertices)
- $\{53, 40, 28, 16; 1, 4, 10, 28\}$ (2916 vertices)

Nonexistence of Q-polynomial association schemes [GVW21] with parameters listed as feasible by Williford has been shown for

- 29 cases of 3-class primitive Q-polynomial association schemes
  - double counting has been used in two cases
- 92 cases of 4-class Q-bipartite Q-polynomial association schemes
- 11 cases of 5-class Q-bipartite Q-polynomial association schemes

Nonexistence of infinite families

Association schemes with the following parameters do not exist.

- distance-regular graphs with intersection arrays $\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\}$ ($r, t \geq 1$)
- primitive Q-polynomial association schemes with Krein arrays $\{2r^2-1, 2r^2-2, r^2+1; 1, 2, r^2-1\}$ ($r \geq 3$ odd)
- Q-bipartite Q-polynomial association schemes with Krein arrays $\left\{m, m-1, \frac{m(r^2-1)}{r^2}, m-r^2+1; 1, \frac{m}{r^2}, r^2-1, m\right\}$ ($m, r \geq 3$ odd)
- Q-bipartite Q-polynomial association schemes with Krein arrays $\left\{\frac{r^2+1}{2}, \frac{r^2-1}{2}, \frac{(r^2+1)^2}{2r(r+1)}, \frac{(r-1)(r^2+1)}{4r}, \frac{r^2+1}{2r}; 1, \frac{(r-1)(r^2+1)}{2r(r+1)}, \frac{(r+1)(r^2+1)}{4r}, \frac{(r-1)(r^2+1)}{2r}, \frac{r^2+1}{2}\right\}$ ($r \geq 5$, $r \equiv 3 \pmod 4$)
- Q-antipodal Q-polynomial association schemes with Krein arrays $\left\{r^2-4, r^2-9, \frac{12(s-1)}{s}, 1; 1, \frac{12}{s}, r^2-9, r^2-4\right\}$ ($r \geq 5, s \geq 4$)
  - Corollary: a tight 4-design in $H((9a^2+1)/5, 6)$ does not exist [GSV20].

Using Schönberg's theorem

- Schönberg's theorem: A polynomial $f : [-1, 1] \to \mathbb{R}$ of degree D is positive definite on S^{m-1} iff it is a nonnegative linear combination of Gegenbauer polynomials Q^m_ℓ ($0 \leq \ell \leq D$).
- Theorem (Kodalen, Martin): If $(X, \mathcal{R})$ is an association scheme, then

$$Q^{m_i}_\ell\left(\frac{1}{m_i} L^*_i\right) = \frac{1}{|X|} \sum_{j=0}^{D} \theta_{\ell j} L^*_j$$

for some nonnegative constants $\theta_{\ell j}$ ($0 \leq j \leq D$), where $m_i = \operatorname{rank}(E_i)$ and $L^*_i = (q^h_{ij})^D_{h,j=0}$.

[23]:
```
q594 = drg.QPolyParameters([9, 8, 81/11, 63/8], [1, 18/11, 9/8, 9])
q594.order()
```

[23]:
```
594
```

[24]:
```
q594.check_schoenberg()
```

```
...
InfeasibleError: Gegenbauer polynomial 4 on L*[1] not nonnegative: nonexistence
 by Kodalen19, Corollary 3.8.
```

The Terwilliger polynomial

- Terwilliger has observed that for a Q-polynomial distance-regular graph Γ, there exists a polynomial T of degree 4 whose coefficients can be expressed in terms of the intersection numbers of Γ such that $T(\theta) \geq 0$ for each non-principal eigenvalue θ of the local graph at a vertex of Γ.
- sage-drg can be used to compute this polynomial.

[25]:
```
p750 = drg.DRGParameters([49, 40, 22], [1, 5, 28])
p750.order()
```

[25]: 750

[26]:
```
T750 = p750.terwilligerPolynomial()
T750
```

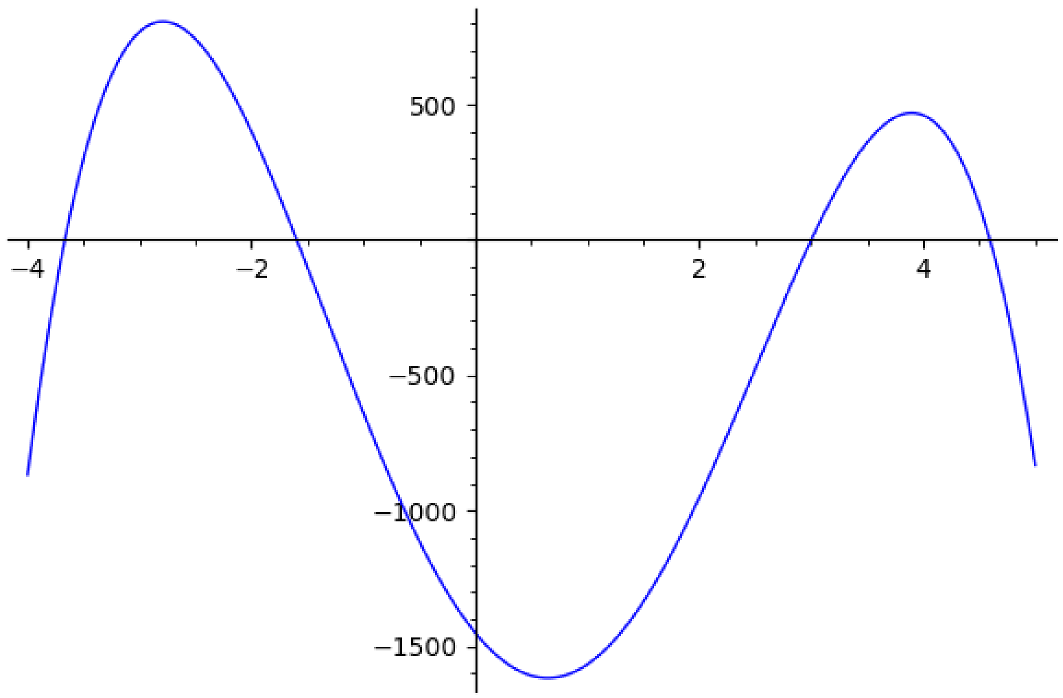
[26]: $-18\,x^4 + 42\,x^3 + 366\,x^2 - 506\,x - 1452$

[27]:
```
sorted(s.rhs() for s in solve(T750 == 0, x))
```

[27]: $\left[-\frac{11}{3}, -\frac{1}{6}\sqrt{345} + \frac{3}{2}, 3, \frac{1}{6}\sqrt{345} + \frac{3}{2}\right]$

[28]:
```
plot(T750, (x, -4, 5))
```

[28]:



We may now use [BCN, Thm. 4.4.4] to further restrict the possible non-principal eigenvalues of the local graphs.

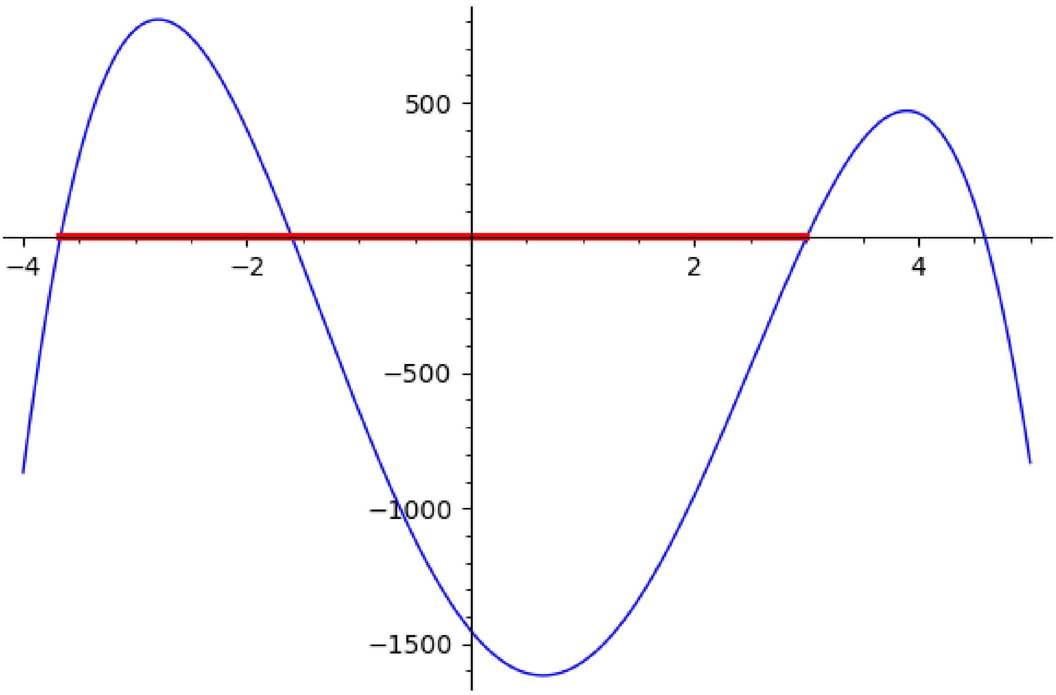
[29]:
```
l, u = -1 - p750.b[1] / (p750.theta[1] + 1), -1 - p750.b[1] / (p750.theta[3]␣
↪+ 1)
```

```
l, u
```

[29]: $\left(-\frac{11}{3}, 3\right)$

[30]:
```
plot(T750, (x, -4, 5)) + line([(l, 0), (u, 0)], color="red", thickness=3)
```

[30]:



Since graph eigenvalues are algebraic integers and all non-integral eigenvalues of the local graph lie on a subinterval of $(-4, -1)$, it can be shown that the only permissible non-principal eigenvalues are $-3, -2, 3$.

We may now set up a system of equations to determine the multiplicities.

[31]:
```
var("m1 m2 m3")
solve([1 + m1 + m2 + m3 == p750.k[1],
       1 * p750.a[1] + m1 * 3 + m2 * (-2) + m3 * (-3) == 0,
       1 * p750.a[1]^2 + m1 * 3^2 + m2 * (-2)^2 + m3 * (-3)^2 == p750.k[1] *
 ↪p750.a[1]],
      (m1, m2, m3))
```

[31]: $\left[\left[m_1 = \left(\frac{96}{5}\right), m_2 = \left(\frac{104}{5}\right), m_3 = 8\right]\right]$

Since all multiplicities are not nonnegative integers, we conclude that there is no distance-regular graph with intersection array

- $\{49, 40, 22; 1, 5, 28\}$ (750 vertices)
- $\{109, 80, 22; 1, 10, 88\}$ (1200 vertices)
- $\{164, 121, 33; 1, 11, 132\}$ (2420 vertices)

Distance-regular graphs with classical parameters

We use a similar technique to prove nonexistence of certain distance-regular graphs with classical parameters (D, b, α, β):

- $(3, 2, 2, 9)$ (430 vertices)
- $(3, 2, 5, 21)$ (1100 vertices)
- $(6, 2, 2, 107)$ (87 725 820 468 vertices)
- $(b, \alpha) = (2, 1)$ and
  - $D = 4$, $\beta \in \{8, 10, 12\}$
  - $D = 5$, $\beta \in \{16, 17, 19, 20, 21, 28\}$
  - $D = 6$, $\beta \in \{32, 33, 34, 35, 36, 38, 40, 46, 49, 54, 60\}$
  - $D = 7$, $\beta \in \{64, 65, 66, 67, 69, 70, 71, 72, 73, 74, 77, 79, 81, 84, 85, 92, 99, 124\}$
  - $D = 8$, $\beta \in \{128, 129, 130, 131, 133, 134, 135, 136, 137, 139, 140, 141, 151, 152, 155, 158, 160, 165, 168, 174, 175, 183, 184, 190, 202, 238, 252\}$
  - $D \geq 3$, $\beta \in \{2^{D-1}, 2^D - 4\}$

Local graphs with at most four eigenvalues

- Lemma (Van Dam): A connected graph on n vertices with spectrum

  $$\theta_0{}^{\ell_0} \quad \theta_1{}^{\ell_1} \quad \theta_2{}^{\ell_2} \quad \theta_3{}^{\ell_3}$$

  is walk-regular with precisely

  $$w_r = \frac{1}{n} \sum_{i=0}^{3} \ell_i \cdot (\theta_i)^r$$

  closed r-walks ($r \geq 3$) through each vertex.
  - If r is odd, w_r must be even.
- A distance-regular graph Γ with classical parameters $(D, 2, 1, \beta)$ has local graphs with
  - precisely three distinct eigenvalues if $\beta = 2^D - 1$, and then Γ is a bilinear forms graph (Gavrilyuk, Koolen)
  - precisely four distinct eigenvalues if $(\beta + 1) \mid (2^D - 2)(2^D - 1)$, and then $\beta = 2^D - 2$ (or w_3 is nonintegral)
- There is no distance-regular graph with classical parameters $(D, 2, 1, \beta)$ such that
  - $(D, \beta) \in \{(3, 5), (4, 9), (4, 13), (5, 29), (6, 41), (6, 61), (7, 125), (8, 169), (8, 253)\}$
  - $D \geq 3$ and $\beta = 2^D - 3$