HMS symmetries and hypergeometric systems

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June 21, 2021

Hilbert's 21st problem

Show that there exists a linear differential equation of the Fuchsian class with given singular points and monodromy group.

21. Beweis der Existenz linearer Differentialgleichungen mit vorgeschriebener Monodromiegruppe.

Aus der Theorie der linearen Differentialgleichungen mit einer unabhängigen Veränderlichen z möchte ich auf ein wichtiges Problem hinweisen, welches wohl bereits Riemann im Sinne gehabt hat, und welches darin besteht, zu zeigen, daß es stets *eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulären Stellen und einer gegebenen Monodromiegruppe giebt.* Die Aufgabe verlangt also die Auffindung von n Functionen der Variabeln z, die sich überall in der complexen z-Ebene regulär verhalten, außer etwa in den gegebenen singulären Stellen: in diesen dürfen sie nur von endlich hoher Ordnung unendlich werden und beim Umlauf der Variabeln z um dieselben erfahren sie die gegebenen linearen Substitutionen. Die Existenz solcher Differentialgleichungen ist durch Constantenzählung wahrscheinlich gemacht worden, doch gelang der strenge Beweis bisher nur in dem besonderen Falle, wo die Wurzeln der Fundamentalgleichungen der gegebenen Substitutionen sämtlich vom absoluten Betrage 1 sind. Diesen Beweis hat L. Schlesinger[1]) auf Grund der Poincaréschen Theorie der Fuchsschen ζ-Functionen erbracht. Es würde offenbar die Theorie der linearen Differentialgleichungen ein wesentlich abgeschlosseneres Bild zeigen, wenn die allgemeine Erledigung des bezeichneten Problems gelänge.

*Mathematische Probleme / Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse / Zeitschriftenband (1900) / Zeitschriftenheft / Artikel / 263 - 297

Fuchsian type

System of linear differential equations of **Fuchsian** type:

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = A(z) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$A(z)$ holomorphic on $\bar{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}$, with a pole of order 1 at a_j, $1 \leq j \leq N$.

In particular, $\sum_{i=0}^{n} q_i(z) y^{(n-i)} = 0$, $q_n(z) = 1$, is Fuchsian if and only if $q_i(z)(z-a)^i$ is holomorphic at $z = a$ for $a \in \mathbb{C}$ and $q_i(z) z^i$ is holomorphic at $z = \infty$, for $0 \leq i \leq n$.

Monodromy

$\dot{y} = A(z)y$, $\{a_1, \ldots, a_N\}$ singular points
γ closed path in $\bar{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}$

$y_1, \ldots, y_n$ basis of solutions of the system
$\tilde{y}_1, \ldots, \tilde{y}_n$ analytic continuations of $y_1, \ldots, y_n$ along γ

Monodromy

*https://www.minimalistjourneys.com/lake-bled-24-hours-itinerary/

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$(\tilde{y}_1, \ldots, \tilde{y}_n) = \rho_\gamma(y_1, \ldots, y_n)$, $\rho_\gamma \in \mathrm{GL}_n(\mathbb{C})$

$\rho : \pi_1(\bar{\mathbb{C}}\setminus\{a_1, \ldots, a_N\}) \to \mathrm{GL}_n(\mathbb{C}),\ [\gamma] \mapsto \rho_\gamma$ **monodromy** representation

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$zy' - \alpha y = 0$
$\gamma(t) = e^{2\pi \mathrm{i} t}$
$y = z^\alpha$
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$$\rho : \pi_1(\bar{\mathbb{C}} \setminus \{0, \infty\}) \cong \mathbb{Z} \to \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*,\ k \mapsto e^{2\pi \mathrm{i} k\alpha}$$

Hilbert's 21st problem

Formulation

- $\{a_1, \ldots, a_N\} \subset \bar{\mathbb{C}}$
- $\rho : \pi_1(\bar{\mathbb{C}} \setminus \{a_1, \ldots, a_N\}) \to \mathrm{GL}_n(\mathbb{C})$ a homomorphism

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Does there exist a system of linear differential equations of Fuchsian type with the monodromy representation equal to ρ?

Hilbert's 21st problem

Progress

Solution.

Plemelj, 1908

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*https://mathshistory.st-andrews.ac.uk/Biographies/Plemelj/

*https://www.uni-lj.si/o_univerzi_v_ljubljani/100_let_ul/

Hilbert's 21st problem

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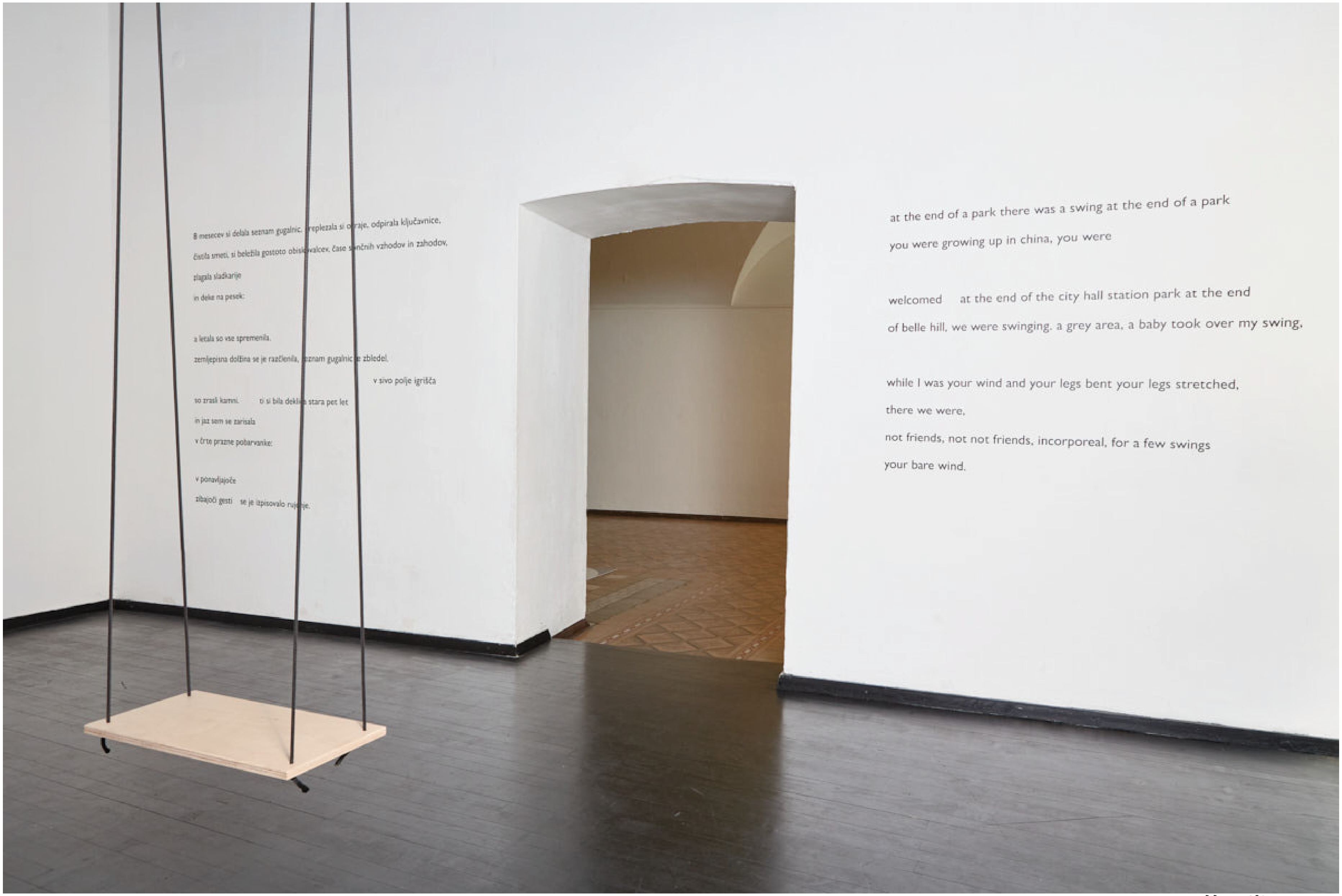
Plemelj, 1908

Plemelj's proof is not correct.

Il'yashenko, ∼1970

Counterexample.

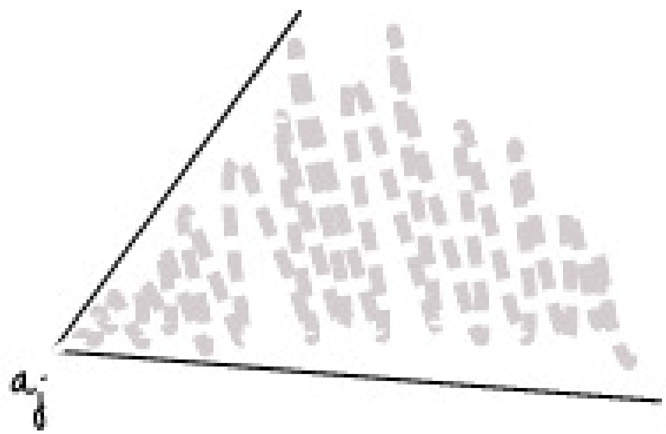
Bolibrukh, 1989



Kristina Hočevar: Rujenje / Half of a C / C 的一半, 2021

*curator: Iza Pevec, photo: Simao Bessa © Škuc Gallery

Regular singularities



punctured angular sector

a_j finite: $|y_i(z)| = O(|z - a_j|^{-m})$ for some $m \geq 0$ as $z \to a_j$,
$a_j = \infty$: $|y_i(z)| = O(|z|^m)$ for some $m \geq 0$ as $z \to \infty$.

It also has an algebraic interpretation, generalises to higher dimensions.

Riemann-Hilbert correspondence I

X complex manifold, $D \subset X$ divisor

There is 1:1 correspondence:

$$\{\text{sys. of lin. diff. eq. on } X \text{ with reg. sing. } \subset D\} \longleftrightarrow \mathrm{rep}(\pi_1(X \backslash D)).$$

Deligne 1970

Riemann-Hilbert correspondence I

There are equivalences of abelian categories:

$$\begin{array}{ccc} \{\text{sys. of lin. diff. eq. on } X \text{ with reg. sing.} \subset D\} & \xrightarrow{\sim} & \mathrm{rep}(\pi_1(X \setminus D)) \\ \downarrow \wr & & \uparrow \wr \\ \{\text{sys. of lin. diff. eq. on } X \setminus D \text{ with no sing.}\} & \xrightarrow[\sim]{} & \mathrm{Loc}(X \setminus D) \end{array}$$

$\mathrm{Loc}(X \setminus D)$ the category of **local systems** on $X \setminus D$, i.e. locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on $X \setminus D$

Drago Tršar: Manifestanti I, 1959

*photo: Zala Lenarčič

D-modules

$X \subset \mathbb{C}^n$ open set

$\mathcal{O}(X)$ holomorphic functions globally defined on X

D partial differential operators with coefficients in $\mathcal{O}(X)$

$D = \{\sum_{i_1,\ldots,i_n} f_{i_1\ldots i_n}(\frac{\partial}{\partial x_1})^{i_1}\cdots(\frac{\partial}{\partial x_n})^{i_n} \mid f_{i_1\ldots i_n} \in \mathcal{O}(X)\}$

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$\{u \in \mathcal{O}(X) \mid Pu = 0\} \cong \mathrm{Hom}_D(D/DP, \mathcal{O}(X))$

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If $\mathcal{M}$ **"holonomic"**, then $\mathcal{F}_i := \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ is **constructible**, $i \in \mathbb{N}$,
i.e. $\exists$ stratification $X = \sqcup_\alpha X_\alpha$ s.t. $\mathcal{F}_i|_{X_\alpha}$ local system.

Rok Horvat: A line on the Earth is a vector in a cloud, 2021

Derived category

$\mathcal{A}$ abelian category (e.g. $\mathrm{Mod}(\mathcal{D}_X)$, $\mathrm{Mod}(\mathbb{C}_X)$)

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$F = \mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X) : \mathrm{Mod}(\mathcal{D}_X)^\circ \to \mathrm{Mod}(\mathbb{C}_X)$
$RF = R\,\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X) : D^-(\mathcal{D}_X)^\circ \to D^+(\mathbb{C}_X)$
$H^i(RF) = \mathcal{E}xt^i_{\mathcal{D}_X}(-, \mathcal{O}_X)$

Suzana Brborović: Idea of the Ideal, 2015

*acrylic and ink on canvas, 200 x 200 cm

Riemann-Hilbert correspondence

$D_{rh}(\mathcal{D}_X) \subset D(\mathcal{D}_X)$ complexes with *regular, holonomic* cohomology
$D_c(\mathbb{C}_X) \subset D(\mathbb{C}_X)$ complexes with *constructible* cohomology

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There is an anti-equivalence of triangulated categories

$$R\,\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X) : D^b_{rh}(\mathcal{D}_X)^\circ \xrightarrow{\sim} D^b_c(\mathbb{C}_X).$$

Kashiwara 1980; Mebkhout 1984; Beilinson, Bernstein 1981

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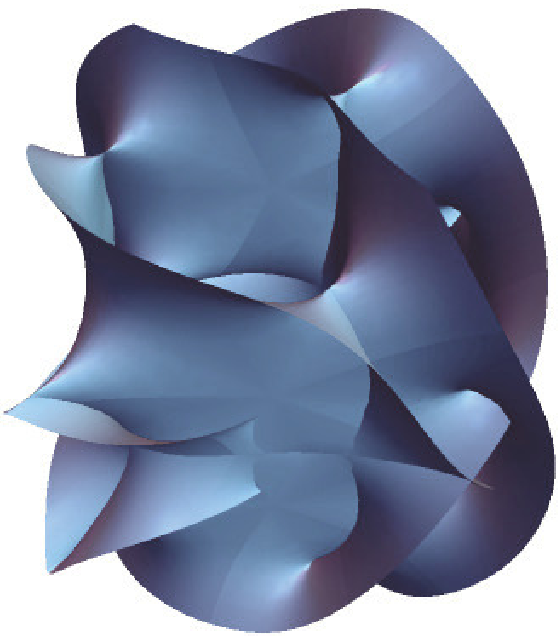
$\mathrm{Perv}(X) := R\,\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X)(\operatorname{mod} \mathcal{D}_X^\circ)[\dim X]$ **perverse sheaves**

Metka Krašovec: Triple mirror, 1992

Mirror symmetry



Spaces in string theory have **complex** and **symplectic** structure.

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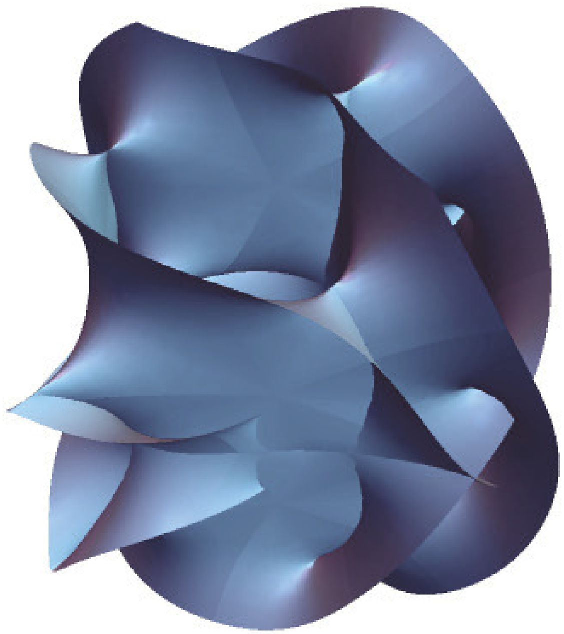

Spaces in string theory have **complex** and **symplectic** structure.

The spaces come in "mirror" pairs X, X^o, the complex and symplectic structure are interlaced. The complex geometry of X mirrors the symplectic geometry of its mirror X^o, and vice versa.

Homological mirror symmetry

$\mathcal{K}_X$ = space of complex structures on X^o

Conjecture

$$\pi_1(\mathcal{K}_X) \curvearrowright D^b(X).$$

Kontsevich 1994

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Corollary

$$\pi_1(\mathcal{K}_X) \curvearrowright K_0(X)_{\mathbb{C}}.$$

Gabrijel Stupica: Studio, 1962

Conifold

$Y = \mathrm{Spec}(\mathbb{C}[x, y, z, u]/(xz - yu))$ conifold (=cone over $\mathbb{P}^1 \times \mathbb{P}^1$)

$X = \mathrm{Bl}_{(x,y)} Y$ a small resolution of Y

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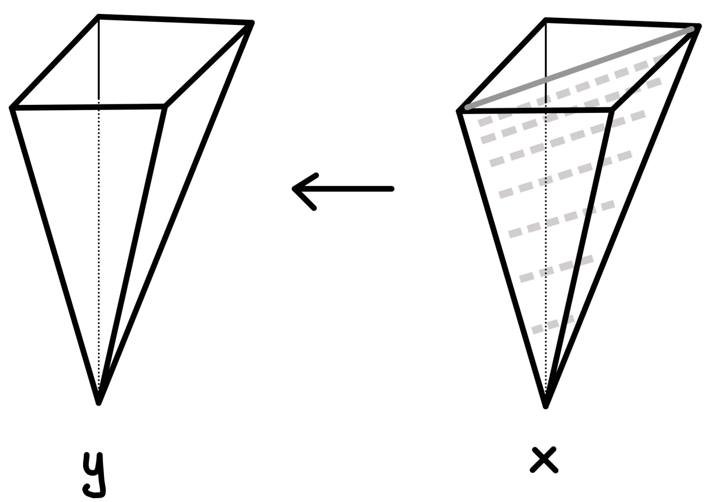

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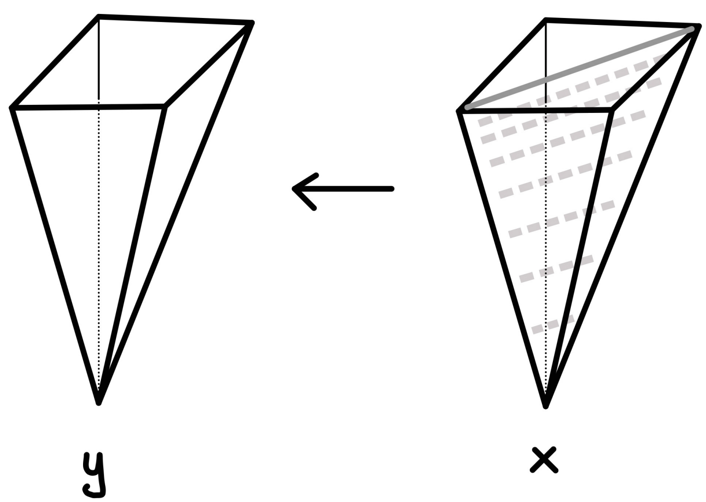

$\mathbb{C}^* \curvearrowright \mathbb{C}^4,\ t.(x_1, x_2, x_3, x_4) = (t^{-1}x_1, t^{-1}x_2, tx_3, tx_4)$

$Y \cong \mathbb{C}^4 /\!\!/ \mathbb{C}^*$ $(= \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{C}^*}))$

$X \cong (\mathbb{C}^4 \setminus V(x_1, x_2)) /\!\!/ \mathbb{C}^*$

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$D^b(X) \hookrightarrow D^b([\mathbb{C}^4/\mathbb{C}^*])$, image generated by $\mathcal{O}_{\mathbb{C}^4}, \mathcal{O}_{\mathbb{C}^4} \otimes V(1)$ ($V(n)$ is $\mathbb{C}^*$-representation with character n, i.e. $t.x = t^n x$).

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$\gamma_0, \gamma_1, \gamma_\infty \in \pi_1(\mathcal{K}_X) \curvearrowright D^b(X)$:

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

Joni Zakonjšek: Do you still remember love? 2007-2010

*photo: Kristijan Robič, archive KID KIBLA, from the exposition Joni Zakonjšek: Blaženo tvoja / Blissfully yours, Gallery KiBela, MMC KIBLA, 2017

The space of complex structures on X^o

Toric varieties

$(\mathbb{C}^*)^n \curvearrowright \mathbb{C}^d$ unimodular representation

$X = (\mathbb{C}^d \setminus (\mathbb{C}^d)_{ns}) /\!\!/ (\mathbb{C}^*)^n$ affine Gorenstein toric variety

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Kite 2017

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X a small resolution of the conifold
$(1/2\pi \mathrm{i}) \log \mathcal{K}_X = (1/2\pi \mathrm{i}) \log(\mathbb{C} \setminus \{0, 1\}) = \mathbb{C} \setminus \mathbb{Z}$

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Halpern-Leistner, Sam 2016

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Halpern-Leistner, Sam 2016

The statement can be generalised to reductive groups (with $X(G)^W \neq 0$ and up to some "genericity" assumptions).

Zdenka Badovinac: Bigger than Myself - Heroic voices from ex-Yugoslavia, MAXXI Rome, 2021

Conifold

Gauss hypergeometric equation

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$\pi_1(\mathcal{K}_X) \curvearrowright K_0(X)_{\mathbb{C}}$ corresponds to the diff. eq. $z(1-z)y'' - zy' = 0$.

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$(\mathbb{C}^4)^* \curvearrowright \mathbb{C}^4$ coordinate-wise

$\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^4,\ t \mapsto (t^{-1}, t^{-1}, t, t)$

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To get an action for other a, b, c we need to replace $D^b(X)$ by a bigger category $\tilde{D}$ such that $X((\mathbb{C}^*)^3)$ acts on it.

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$K_0(\tilde{D})_{\mathbb{C}}$ is then a module over $\mathbb{C}[X((\mathbb{C}^*)^3)] \cong \mathbb{C}[(\mathbb{C}^*)^3]$, specialising at $h \in (\mathbb{C}^*)^3$ gives Gauss hypergeometric equation with parameters $\sim \log h$.

GKZ hypergeometric systems

The hypergeometric function ${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$ satisfies the Gauss hypergeometric equation.

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The multidimensional hypergeometric series

$$\Phi_\gamma(z_1, \ldots, z_d) = \sum_{l \in L} \prod_{i=1}^{d} \frac{z_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)},$$

where L is a sublattice of rank n in $\mathbb{Z}^d$, satisfies a **GKZ hypergeometric system of differential equations** (depending on L, γ).

Gelfand, Kapranov, Zelevinsky 1990

Categorical symmetries

System of differential equations

$\pi_1(\mathcal{K}_X) \curvearrowright K_0(X)_{\mathbb{C}}$ gives the monodromy of a GKZ hypergeometric system (for generic parameters).

Š., Van den Bergh 2020

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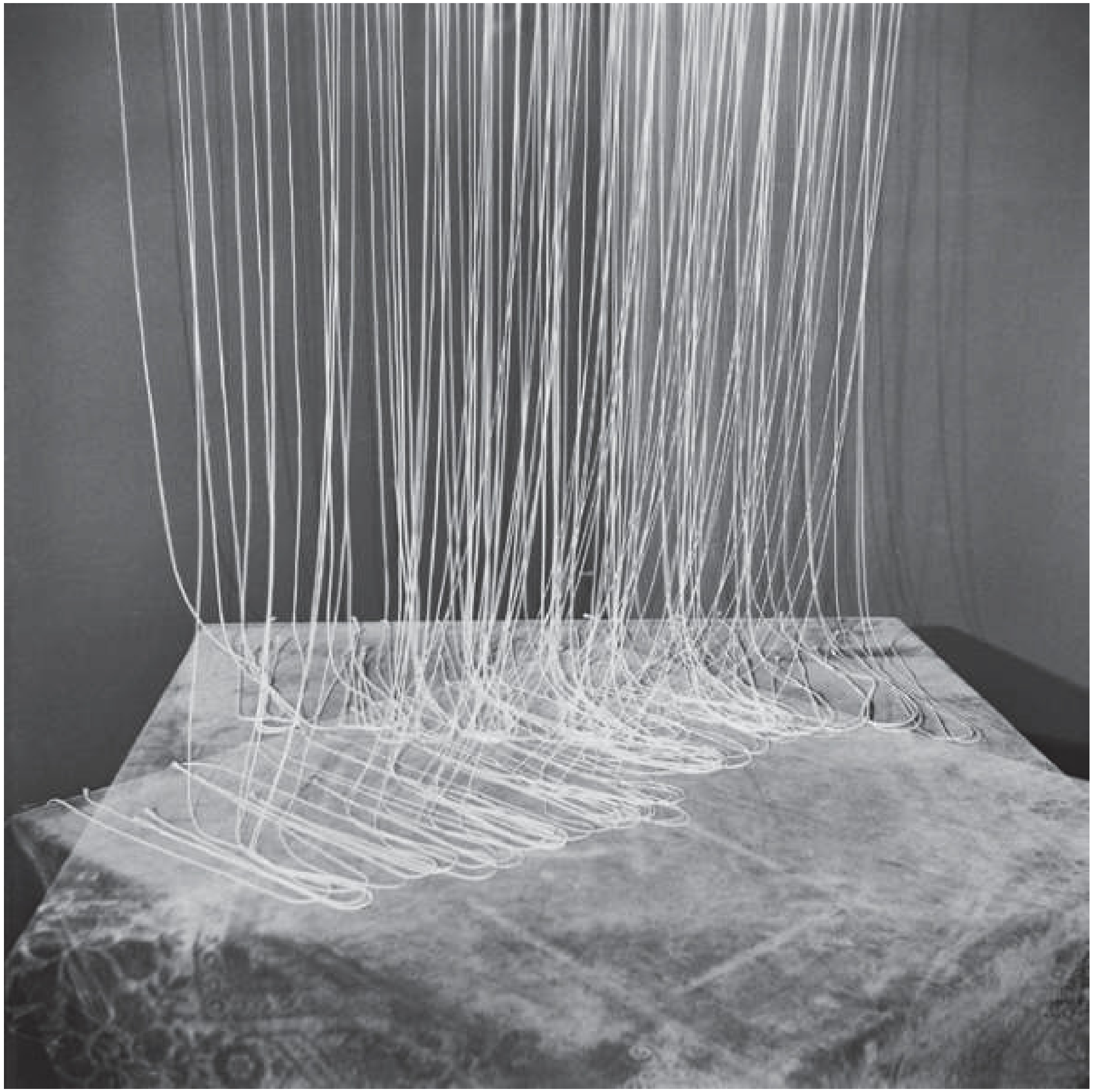
More precisely, there exists $\pi_1(\mathcal{K}_X) \curvearrowright \tilde{D}$ such that $\pi_1(\mathcal{K}_X) \curvearrowright K_0(\tilde{D})_{\mathbb{C}}$ after specialisation gives the monodromy of a GKZ hypergeometric system (L is determined by $(\mathbb{C}^*)^n \curvearrowright \mathbb{C}^d$, parameters γ_i by specialisation).



Vlado Stjepić, 1980-1983

Liftings

$\pi_1(\mathcal{K}_X) \curvearrowright D^b(X)$, i.e. a "local system of categories" on $\mathcal{K}_X$, extends to a "perverse sheaf of categories" on $(\mathbb{C}^*)^n$, a "perverse schober".

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$\pi_1(\mathcal{K}_X) \curvearrowright D^b(X)$ beyond quasi-symmetric case is in progress...

Hvala

Ivo Mršnik, pencil, frottage on computer printout paper, 2009

*photo: Sašo Kovačič