MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Structure-Preserving Interpolation for Bilinear Systems

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June 21 – 25, 2021

8th European Congress of Mathematics, Portorož, Slovenia
Virtual Conference

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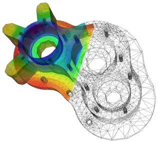
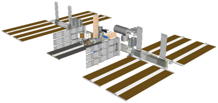
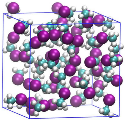
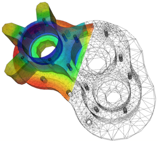
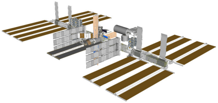
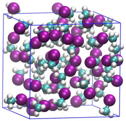
  

Real-world modeling via PDEs

$$\partial_t v = \Delta_x v, \quad \partial_t v + v \nabla_x v = \mu \Delta_x v$$

Real-world modeling via PDEs

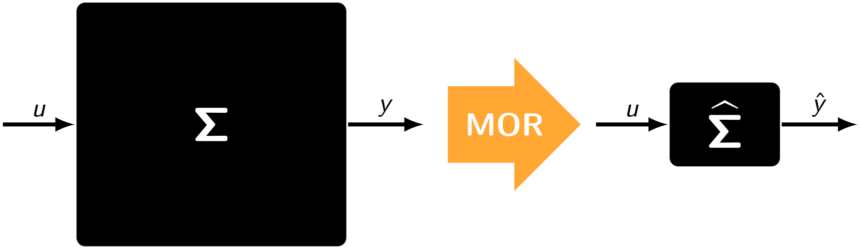
$$\partial_t v = \Delta_x v, \quad \partial_t v + v \nabla_x v = \mu \Delta_x v$$

Spatial discretization

Dynamical input-output systems

$$u \longrightarrow \boxed{\Sigma : \begin{cases} \dot{x}(t) = f(t, x, u), \\ y(t) = g(t, x, u) \end{cases}} \longrightarrow y$$

with inputs $u \colon \mathbb{R} \to \mathbb{R}$, states $x \colon \mathbb{R} \to \mathbb{R}^n$, outputs $y \colon \mathbb{R} \to \mathbb{R}$, and $n \gtrapprox 10^6$ large.

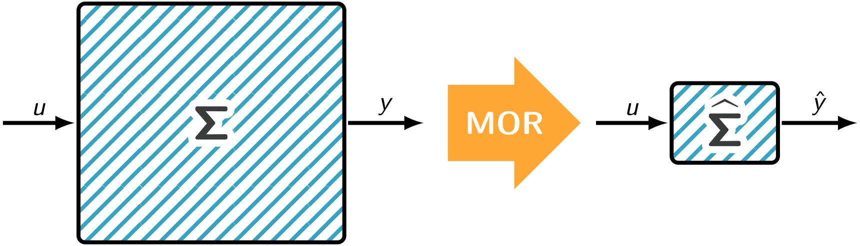


Model order reduction

Approximate the input-to-output behavior,

$$\|y - \hat{y}\| \leq \mathrm{tol} \cdot \|u\|,$$

while reducing the number of equations.



Structure-preserving model order reduction

Approximate the input-to-output behavior,

$$\|y - \hat{y}\| \leq \mathrm{tol} \cdot \|u\|,$$

while reducing the number of equations **and preserve the internal structure**.

Structured Linear Systems

Classical Systems

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

$$G(s) = C(sI_n - A)^{-1}B$$

Mechanical Systems

$$\begin{aligned}M\ddot{x}(t) + E\dot{x}(t) + Kx(t) &= B_{\mathrm{u}}u(t),\\ y(t) &= C_{\mathrm{p}}x(t) + C_{\mathrm{v}}\dot{x}(t)\end{aligned}$$

$$G(s) = (C_{\mathrm{p}} + sC_{\mathrm{v}})(s^2M + sE + K)^{-1}B_{\mathrm{u}}$$

Time-Delay Systems

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_{\mathrm{d}}x(t-\tau) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

$$G(s) = C(sI_n - A - e^{-s\tau}A_{\mathrm{d}})^{-1}B$$

...

Structured **Bilinear** Systems

Classical Systems

$$\dot{x}(t) = Ax(t) + \boldsymbol{Nx(t)u(t)} + Bu(t),$$
$$y(t) = Cx(t)$$

Mechanical Systems

$$M\ddot{x}(t) + E\dot{x}(t) + Kx(t) = \boldsymbol{N_{\mathrm{p}}x(t)u(t)} + \boldsymbol{N_{\mathrm{v}}\dot{x}(t)u(t)} + B_{\mathrm{u}}u(t),$$
$$y(t) = C_{\mathrm{p}}x(t) + C_{\mathrm{v}}\dot{x}(t)$$

Time-Delay Systems

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + \boldsymbol{Nx(t)u(t)} + Bu(t),$$
$$y(t) = Cx(t)$$

...

Linear systems in the frequency domain

$$G(s) = \mathcal{C}(s)\,\mathcal{K}(s)^{-1}\,\mathcal{B}(s)$$

with matrix functions

$$\mathcal{C}\colon \mathbb{C} \to \mathbb{C}^{1 \times n},$$
$$\mathcal{K}\colon \mathbb{C} \to \mathbb{C}^{n \times n},$$
$$\mathcal{B}\colon \mathbb{C} \to \mathbb{C}^{n \times 1}.$$

Linear systems in the frequency domain

$$G(s) = \mathcal{C}(s)\,\mathcal{K}(s)^{-1}\,\mathcal{B}(s)$$

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$$\mathcal{C}\colon \mathbb{C} \to \mathbb{C}^{1\times n},$$
$$\mathcal{K}\colon \mathbb{C} \to \mathbb{C}^{n\times n},$$
$$\mathcal{B}\colon \mathbb{C} \to \mathbb{C}^{n\times 1}.$$

MOR by projection

$$\widehat{G}(s) = \widehat{\mathcal{C}}(s)\,\widehat{\mathcal{K}}(s)^{-1}\,\widehat{\mathcal{B}}(s)$$

for $V, W \in \mathbb{C}^{n\times r}$ s.t.

$$\widehat{\mathcal{C}}(s) = \mathcal{C}(s)V,$$
$$\widehat{\mathcal{K}}(s) = W^{\mathsf{H}}\mathcal{K}(s)V,$$
$$\widehat{\mathcal{B}}(s) = W^{\mathsf{H}}\mathcal{B}(s).$$

Problem: How to choose V and W?

Transfer function interpolation: Find V and W such that

$$G(\sigma_j) = \widehat{G}(\sigma_j), \qquad j = 1, \ldots, k,$$

holds for given interpolation points $\sigma_1, \ldots, \sigma_k \in \mathbb{C}$.

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$$G(\sigma_j) = \widehat{G}(\sigma_j), \qquad j = 1, \ldots, k,$$

holds for given interpolation points $\sigma_1, \ldots, \sigma_k \in \mathbb{C}$.

Theorem (Structured interpolation) [Beattie/Gugercin '09]

Let $\sigma, \varsigma \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}$, $\mathcal{B}$ and $\widehat{\mathcal{K}}^{-1}$ exist.

1. If $\mathcal{K}(\sigma)^{-1}\mathcal{B}(\sigma) \in \operatorname{span}(V)$ then $G(\sigma) = \widehat{G}(\sigma)$.
2. If $\mathcal{K}(\varsigma)^{-\mathsf{H}}\mathcal{C}(\varsigma)^{\mathsf{H}} \in \operatorname{span}(W)$ then $G(\varsigma) = \widehat{G}(\varsigma)$.

Remark: For $\sigma = \varsigma$ and using 1. and 2., it follows that $\frac{\partial}{\partial s}G(\sigma) = \frac{\partial}{\partial s}\widehat{G}(\sigma)$.

Classical bilinear systems

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \quad y(t) = Cx(t)$$

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Volterra Series Expansion

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} g_k(t_1, \ldots, t_k) \left(u(t - \sum_{i=1}^{j} t_i) \cdots u(t - t_1) \right) \mathrm{d}t_k \cdots \mathrm{d}t_1,$$

with regular Volterra kernels $g_k(t_1, \ldots, t_k) = Ce^{At_k} \left(\prod_{j=1}^{k-1} Ne^{At_{k-j}} \right) B$.

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with regular Volterra kernels $g_k(t_1, \ldots, t_k) = Ce^{At_k} \left(\prod_{j=1}^{k-1} Ne^{At_{k-j}} \right) B$.

Frequency domain: Regular transfer functions

$$G_k(s_1, \ldots, s_k) = C(s_k I_n - A)^{-1} \left(\prod_{j=1}^{k-1} N(s_{k-j} I_n - A)^{-1} \right) B$$

Structure-Preserving Interpolation

Bilinear System Case

Structured regular subsystem transfer functions [Benner/Gugercin/W. '21]

$$G_k(s_1, \ldots, s_k) = \mathcal{C}(s_k)\mathcal{K}(s_k)^{-1} \left(\prod_{j=1}^{k-1} \mathcal{N}(s_{k-j})\mathcal{K}(s_{k-j})^{-1} \right) \mathcal{B}(s_1)$$

Structured regular subsystem transfer functions [Benner/Gugercin/W. '21]

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Resembling sequence of coupled linear systems:

$$G_1(s_1) = \mathcal{C}(s_1)\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1),$$

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Resembling sequence of coupled linear systems:

$$\begin{aligned} G_1(s_1) &= \mathcal{C}(s_1)\boldsymbol{\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1)},\\ G_2(s_1, s_2) &= \mathcal{C}(s_2)\mathcal{K}(s_2)^{-1}\mathcal{N}(s_1)\boldsymbol{\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1)}, \end{aligned}$$

Structured regular subsystem transfer functions [Benner/Gugercin/W. '21]

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Resembling sequence of coupled linear systems:

$$\begin{aligned} G_1(s_1) &= \mathcal{C}(s_1)\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1), \\ G_2(s_1, s_2) &= \mathcal{C}(s_2)\mathcal{K}(s_2)^{-1}\mathcal{N}(s_1)\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1), \\ G_3(s_1, s_2, s_3) &= \mathcal{C}(s_3)\mathcal{K}(s_3)^{-1}\mathcal{N}(s_2)\mathcal{K}(s_2)^{-1}\mathcal{N}(s_1)\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1), \\ &\vdots \end{aligned}$$

Structured regular subsystem transfer functions [Benner/Gugercin/W. '21]

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New: Matrix function for bilinear system parts

$$\boldsymbol{\mathcal{N}} \colon \mathbb{C} \to \mathbb{C}^{n \times n} \quad \textit{and} \quad \widehat{\boldsymbol{\mathcal{N}}}\boldsymbol{(s)} := W^{\mathsf{H}}\boldsymbol{\mathcal{N}(s)}V.$$

Example: Bilinear mechanical systems

Time domain:

$$\begin{aligned} M\ddot{x}(t) + E\dot{x}(t) + Kx(t) &= N_{\mathrm{p}}x(t)u(t) + N_{\mathrm{v}}\dot{x}(t)u(t) + B_{\mathrm{u}}u(t), \\ y(t) &= C_{\mathrm{p}}x(t) + C_{\mathrm{v}}\dot{x}(t) \end{aligned}$$

Frequency domain:

$$\mathcal{C}(s) = C_{\mathrm{p}} + sC_{\mathrm{v}}, \quad \mathcal{K}(s) = s^2M + sE + K, \quad \mathcal{B}(s) = B_{\mathrm{u}}, \quad \mathcal{N}(s) = N_{\mathrm{p}} + sN_{\mathrm{v}}$$

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New problem

Multivariate transfer function interpolation: Find V and W such that

$$G_j(\sigma_1, \ldots, \sigma_j) = \widehat{G}_j(\sigma_1, \ldots, \sigma_j), \qquad j = 1, \ldots, k, \qquad (*)$$

holds for a given sequence of interpolation points $\sigma_1, \ldots, \sigma_k \in \mathbb{C}$.

Theorem (Interpolation by right projection) [Benner/Gugercin/W. '21]

Let $\sigma_1, \ldots, \sigma_k \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist.
Let $\operatorname{span}(V) \supseteq \operatorname{span}([v_1, \ldots, v_k])$, where

$$\begin{aligned} v_1 &= \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \\ v_j &= \mathcal{K}(\sigma_j)^{-1}\mathcal{N}(\sigma_{j-1})v_{j-1}, \qquad 2 \leq j \leq k, \end{aligned}$$

and W of appropriate size, then $(*)$ holds.

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Let $\sigma_1, \ldots, \sigma_k \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(V) \supseteq \operatorname{span}([v_1, \ldots, v_k])$, where

$$v_1 = \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1),$$
$$v_j = \mathcal{K}(\sigma_j)^{-1}\mathcal{N}(\sigma_{j-1})v_{j-1}, \qquad 2 \leq j \leq k,$$

and W of appropriate size, then $(*)$ holds.

Projection space	**Interpolation**
$v_1 = \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1)$	$G_1(\sigma_1) = \widehat{G}_1(\sigma_1)$

Theorem (Interpolation by right projection) [Benner/Gugercin/W. '21]

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$$\begin{aligned} v_1 &= \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \\ v_j &= \mathcal{K}(\sigma_j)^{-1}\mathcal{N}(\sigma_{j-1})v_{j-1}, \qquad 2 \leq j \leq k, \end{aligned}$$

and W of appropriate size, then $(*)$ holds.

Projection space	Interpolation
$v_1 = \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1)$	$G_1(\sigma_1) = \widehat{G}_1(\sigma_1)$
$v_2 = \mathcal{K}(\sigma_2)^{-1}\mathcal{N}(\sigma_1)v_1$	$G_2(\sigma_1, \sigma_2) = \widehat{G}_2(\sigma_1, \sigma_2)$

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and W of appropriate size, then $(*)$ holds.

Projection space	**Interpolation**
$v_1 = \mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1)$	$G_1(\sigma_1) = \widehat{G}_1(\sigma_1)$
$v_2 = \mathcal{K}(\sigma_2)^{-1}\mathcal{N}(\sigma_1)v_1$	$G_2(\sigma_1, \sigma_2) = \widehat{G}_2(\sigma_1, \sigma_2)$
$v_3 = \mathcal{K}(\sigma_3)^{-1}\mathcal{N}(\sigma_2)v_2$	$G_3(\sigma_1, \sigma_2, \sigma_3) = \widehat{G}_3(\sigma_1, \sigma_2, \sigma_3)$

$\vdots$

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(W) \supseteq \operatorname{span}([w_1, \ldots, w_\ell])$, where

$$
\begin{aligned}
w_1 &= \mathcal{K}(\varsigma_\ell)^{-\mathsf{H}} \mathcal{C}(\varsigma_\ell)^{\mathsf{H}}, \\
w_j &= \mathcal{K}(\varsigma_{\ell-j+1})^{-\mathsf{H}} \mathcal{N}(\varsigma_{\ell-j+1})^{\mathsf{H}} w_{j-1}, && 2 \leq j \leq \ell,
\end{aligned}
$$

then $G_1(\varsigma_\ell) = \widehat{G}_1(\varsigma_\ell)$, ..., $G_\ell(\varsigma_1, \ldots, \varsigma_\ell) = \widehat{G}_\ell(\varsigma_1, \ldots, \varsigma_\ell)$ holds.

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(W) \supseteq \operatorname{span}([w_1, \ldots, w_\ell])$, where

Projection space	Interpolation
$w_1 = \mathcal{K}(\varsigma_\ell)^{-\mathsf{H}}\mathcal{C}(\varsigma_\ell)^{\mathsf{H}}$	$G_1(\varsigma_\ell) = \widehat{G}_1(\varsigma_\ell)$

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(W) \supseteq \operatorname{span}([w_1, \ldots, w_\ell])$, where

Projection space	**Interpolation**
$w_1 = \mathcal{K}(\varsigma_\ell)^{-\mathsf{H}}\mathcal{C}(\varsigma_\ell)^{\mathsf{H}}$	$G_1(\varsigma_\ell) = \widehat{G}_1(\varsigma_\ell)$
$w_2 = \mathcal{K}(\varsigma_{\ell-1})^{-\mathsf{H}}\mathcal{N}(\varsigma_{\ell-1})^{\mathsf{H}} w_1$	$G_2(\varsigma_{\ell-1}, \varsigma_\ell) = \widehat{G}_2(\varsigma_{\ell-1}, \varsigma_\ell)$

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(W) \supseteq \operatorname{span}([w_1, \ldots, w_\ell])$, where

Projection space	Interpolation
$w_1 = \mathcal{K}(\varsigma_\ell)^{-\mathsf{H}} \mathcal{C}(\varsigma_\ell)^{\mathsf{H}}$	$G_1(\varsigma_\ell) = \widehat{G}_1(\varsigma_\ell)$
$w_2 = \mathcal{K}(\varsigma_{\ell-1})^{-\mathsf{H}} \mathcal{N}(\varsigma_{\ell-1})^{\mathsf{H}} w_1$	$G_2(\varsigma_{\ell-1}, \varsigma_\ell) = \widehat{G}_2(\varsigma_{\ell-1}, \varsigma_\ell)$
$w_3 = \mathcal{K}(\varsigma_{\ell-2})^{-\mathsf{H}} \mathcal{N}(\varsigma_{\ell-2})^{\mathsf{H}} w_2$	$G_3(\varsigma_{\ell-2}, \varsigma_{\ell-1}, \varsigma_\ell) = \widehat{G}_3(\varsigma_{\ell-2}, \varsigma_{\ell-1}, \varsigma_\ell)$

$\vdots$

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist. Let $\operatorname{span}(W) \supseteq \operatorname{span}([w_1, \ldots, w_\ell])$, where

$$w_1 = \mathcal{K}(\varsigma_\ell)^{-\mathsf{H}} \mathcal{C}(\varsigma_\ell)^{\mathsf{H}},$$
$$w_j = \mathcal{K}(\varsigma_{\ell-j+1})^{-\mathsf{H}} \mathcal{N}(\varsigma_{\ell-j+1})^{\mathsf{H}} w_{j-1}, \qquad 2 \leq j \leq \ell,$$

then $G_1(\varsigma_\ell) = \widehat{G}_1(\varsigma_\ell)$, …, $G_\ell(\varsigma_1, \ldots, \varsigma_\ell) = \widehat{G}_\ell(\varsigma_1, \ldots, \varsigma_\ell)$ holds.

Theorem (Interpolation by two-sided projection) [Benner/Gugercin/W. '21]

Let V and W as before for the interpolation points $\sigma_1, \ldots, \sigma_k, \varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$. Then additionally it holds that

$$G_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell) = \widehat{G}_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell),$$

with $1 \leq p \leq k$ and $1 \leq q \leq \ell$.

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist.

Let span(W) $\supseteq$ span($[w_1, \ldots, w_\ell]$), where

Projection space	Interpolation
w_1, v_1	$G_1(\sigma_1), G_1(\varsigma_\ell)$, $\boldsymbol{G_2(\sigma_1, \varsigma_\ell)}$

$$G_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell) = \widehat{G}_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell),$$

with $1 \leq p \leq k$ and $1 \leq q \leq \ell$.

Theorem (Interpolation by left projection) [Benner/Gugercin/W. '21]

Let $\varsigma_1, \ldots, \varsigma_\ell \in \mathbb{C}$ be interpolation points for which $\mathcal{C}, \mathcal{K}^{-1}, \mathcal{B}$, $\mathcal{N}$ and $\widehat{\mathcal{K}}^{-1}$ exist.
Let span$(W) \supseteq$ span$([w_1, \ldots, w_\ell])$, where

Projection space	Interpolation
w_1, v_1	$G_1(\sigma_1), G_1(\varsigma_\ell)$, **$G_2(\sigma_1, \varsigma_\ell)$**
w_2, v_2	$G_2(\sigma_1, \sigma_2), G_2(\varsigma_{\ell-1}, \varsigma_\ell)$, **$G_3(\sigma_1, \sigma_2, \varsigma_\ell)$**, **$G_3(\sigma_1, \varsigma_{\ell-1}, \varsigma_\ell)$**, **$G_4(\sigma_1, \sigma_2, \varsigma_{\ell-1}, \varsigma_\ell)$**

$\vdots$

$$G_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell) = \widehat{G}_{p+q}(\sigma_1, \ldots, \sigma_p, \varsigma_{\ell-q+1}, \ldots, \varsigma_\ell),$$

with $1 \leq p \leq k$ and $1 \leq q \leq \ell$.

Example: Time-delayed heated rod [Gosea et al. '19]

$$\begin{aligned} Ex(t) &= Ax(t) + A_{\mathrm{d}}x(t-1) + Nx(t)u(t) + Bu(t),\\ y(t) &= Cx(t) \end{aligned}$$

with discretization $n = 5\,000$ and

$$\mathcal{C}(s) = C, \quad \mathcal{K}(s) = sE - A - e^{-s}A_{\mathrm{d}}, \quad \mathcal{B}(s) = B, \quad \mathcal{N}(s) = N.$$

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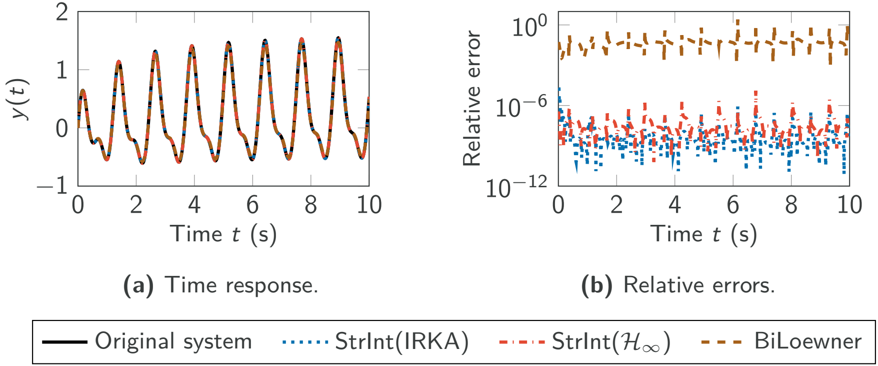
ROM construction:

- bilinear Loewner approximation with 80 complex conjugate points on frequency axis and SVD-based truncation (***BiLoewner***)
- structured interpolation with $\mathcal{H}_\infty$- and TF-IRKA-based points (***StrInt***)
- two-sided approaches, reduced order $r = 8$



(a) Time response.

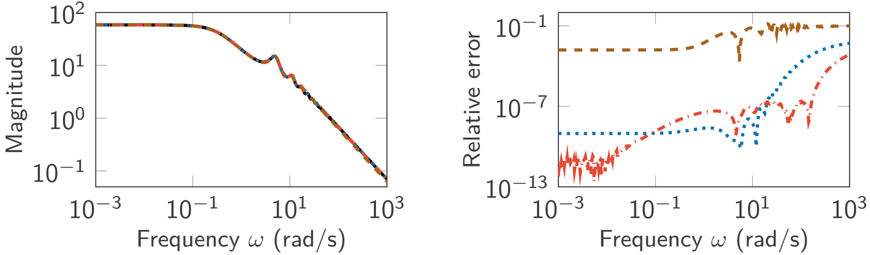
(b) Relative errors.

Input signal:

$$u(t) = \frac{\cos(10t)}{20} + \frac{\cos(5t)}{20}$$

Relative errors:

$$\frac{|y(t) - \hat{y}(t)|}{|\hat{y}(t)|}$$



(a) First subsystem transfer functions.

(b) Relative errors.

Original system · StrInt(IRKA) · StrInt($\mathcal{H}_\infty$) · BiLoewner

Relative errors:

$$\frac{|G_1(\omega \mathrm{i}) - \widehat{G}_1(\omega \mathrm{i})|}{|G_1(\omega \mathrm{i})|}$$

Conclusions & Outlook

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- one- and two-sided projection approaches
- comparison with unstructured approach for bilinear time-delay system

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Further results [Benner/Gugercin/W. '21, '21a]

- matrix interpolation approach for MIMO systems
- implicit and explicit matching of Hermite interpolation conditions
- interpolation of parametric structured bilinear systems

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Further results

[Benner/Gugercin/W. '21, '21a]

- matrix interpolation approach for MIMO systems
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Outlook

- tangential interpolation to handle efficiently MIMO systems
- interpolation of quadratic-bilinear (polynomial) systems

References

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