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Periodic random tilings and non-Hermitian orthogonality

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8th European Congress of Mathematics

Portoroz, Slovenia, 22 June 2021

0 References

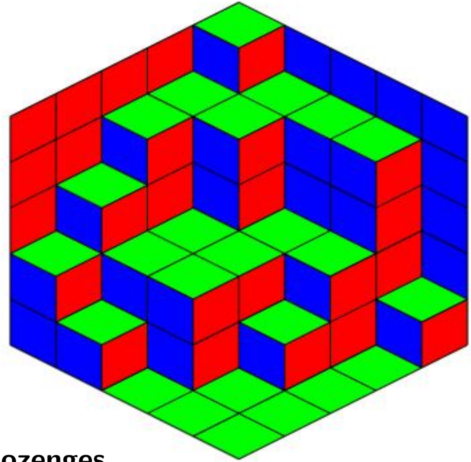
The talk is based on

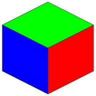
- M. Duits and A.B.J. Kuijlaars,
  The two periodic Aztec diamond and matrix valued orthogonal polynomials, J. Eur. Math. Soc. (2021)
- C. Charlier, M. Duits, A.B.J. Kuijlaars, and J. Lenells,
  A periodic hexagon tiling model and non-Hermitian orthogonal polynomials, Comm. Math. Phys. (2020)
- Alan Groot and Arno B.J. Kuijlaars,
  Matrix valued orthogonal polynomials related to hexagon tilings, preprint arXiv:2104.14822

1. Tiling problems:

Hexagon and Aztec diamond

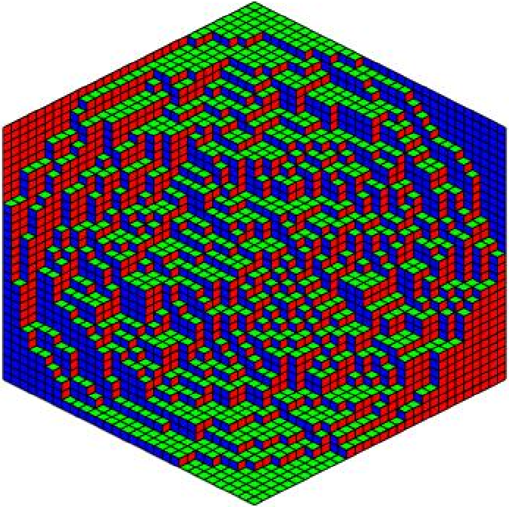
1 Lozenge tiling of a hexagon



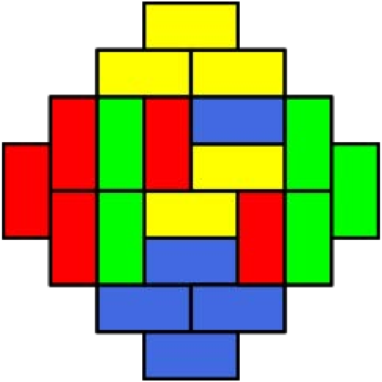


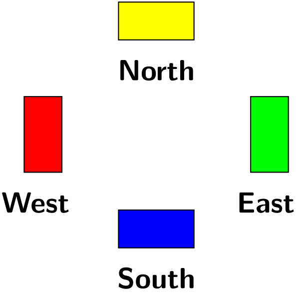
three types of lozenges

1 Arctic circle phenomenon



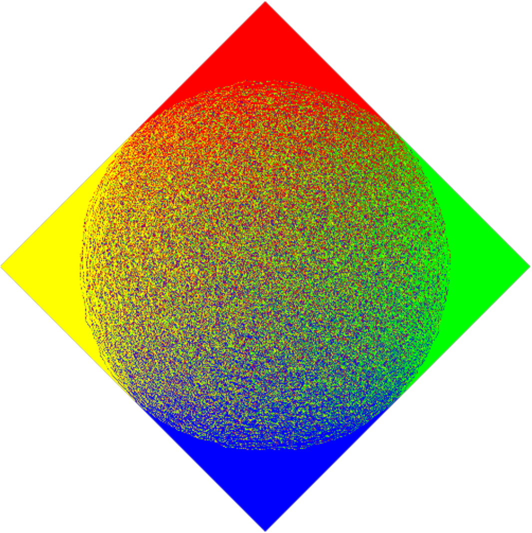
1 Domino tiling of Aztec diamond





- **Tiling with 2×1 and 1×2 rectangles (dominos)**
- **Four types of dominos**

1 Large random tiling: Arctic circle



Deterministic pattern near corners
Solid region
or frozen region

Disorder in the middle
Liquid region

Boundary curve
Arctic circle

1 Some History

Number of domino tilings of Aztec diamond is $2^{N(N+1)/2}$
Elkies, Kuperberg, Larsen, Propp (1992)

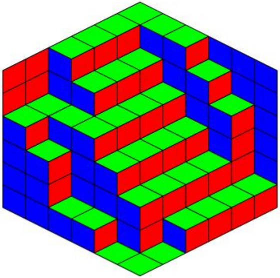
Arctic circle phenomenon Jockush, Propp, Shor (1995)

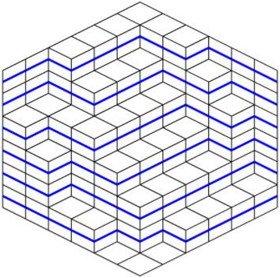
Fluctuations around Arctic circle and connection to random matrix theory (Tracy-Widom distribution) Johansson (2002)

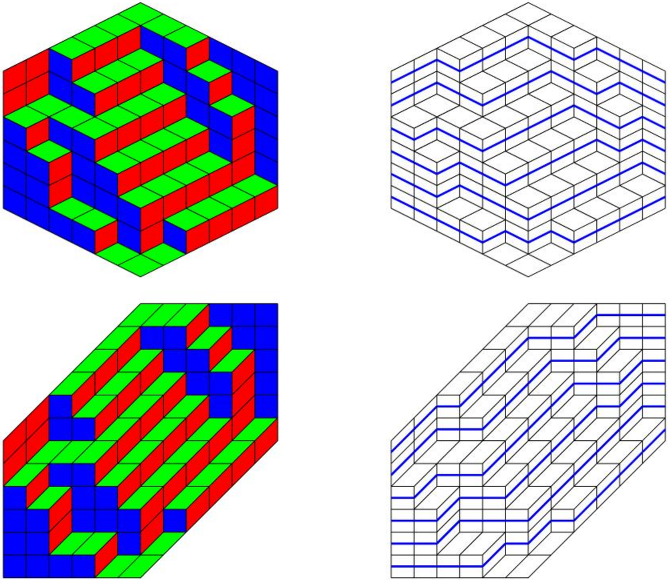
Arctic circle for hexagon tilings
Baik, Kriecherbauer, McLaughlin, Miller (2007) Petrov (2014)

- **Johansson uses Krawtchouk polynomials**
- **Baik et al. use Hahn polynomials**

2. Non-intersecting paths







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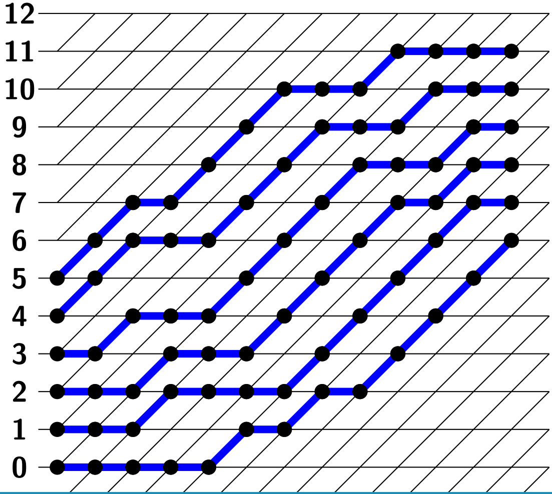
2 Non-intersecting paths on a graph

Paths fit on a graph and give rise to multi level particle system



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2 Determinantal point process

The multi-level particle system is determinantal on discrete state space $\mathcal{X} = \{0, 1, \ldots, L\} \times \mathbb{Z}$

- **There is correlation kernel** $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ **with property that for finite** $\mathcal{A} \subset \mathcal{X}$

$$\det \left[K(\vec{x}, \vec{y}) \right]_{\vec{x}, \vec{y} \in \mathcal{A}} = \text{Prob} \left[\begin{array}{l} \textbf{There is a particle} \\ \textbf{at each } \vec{x} = (m, x) \in \mathcal{A} \end{array} \right]$$

- **Eynard Mehta (1998) give sum formula for the kernel in terms of transition matrices**

$$T_{m', m}(x, y) = \begin{cases} \#\textbf{paths on the lattice} \\ \textbf{from } (m', x) \textbf{ to } (m, y) \end{cases}$$

- **The formula also works in a weighted setting.**

2 Kernel for hexagon of size $N \times M \times (L - M)$

Theorem (Duits K (2021) – very special case)

Correlation kernel has double contour integral formula

$$-\frac{\chi_{m'>m}}{2\pi i} \oint_\gamma (z+1)^{m'-m} z^{y-x} \frac{dz}{z}$$
$$+ \frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma (w+1)^{L-m'} R_N(w,z)(z+1)^m \frac{w^y}{z^x w^{M+N}} \frac{dz dw}{z}$$

where $R_N(w,z) = \sum_{k=0}^{N-1} \frac{p_k(w) p_k(z)}{h_k}$ **is the reproducing kernel**

for orthogonal polynomials on a contour γ **going around** 0

$$\frac{1}{2\pi i} \oint_\gamma p_k(z) p_j(z) \frac{(z+1)^L}{z^{M+N}} dz = h_k \delta_{k,j}$$

2 Non Hermitian orthogonality

$$\frac{1}{2\pi i}\oint_{\gamma} p_k(z)p_j(z)\frac{(z+1)^L}{z^{M+N}}dz = h_k\delta_{k,j}$$

- **Non Hermitian orthogonality on a contour in the complex plane.**
- **The orthogonal polynomials are Jacobi polynomials**

$$p_k(z) \propto P_k^{(-M-N,L)}(2z+1)$$

with one negative parameter (!)

2 Non Hermitian orthogonality

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- **The orthogonal polynomials are Jacobi polynomials**

$$p_k(z) \propto P_k^{(-M-N,L)}(2z+1)$$

with one negative parameter (!)

- **Similar formula applies to the Aztec diamond, but with Jacobi polynomials**

$$p_k(z) \propto P_k^{(-N,N)}(z)$$

3. Weighted tilings

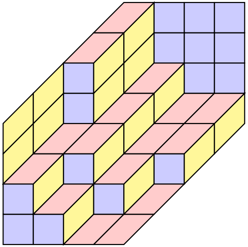
3 Weighted tilings

A weighting on tiles produces a weight on tilings $\mathcal{T}$

$$W(\mathcal{T}) = \prod_{T \in \mathcal{T}} w(T)$$

Probability of a tiling is

$$\textbf{Prob}(\mathcal{T}) = \frac{W(\mathcal{T})}{Z}, \quad Z = \sum_{\mathcal{T}'} W(\mathcal{T}')$$



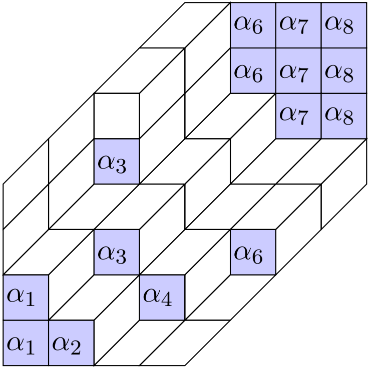
3 Constant weights per column (example)

Weights depend on column

$$w_{\square}(x,y) = \alpha_x,$$

$$w_{\diagup}(x,y) = 1,$$

$$w_{\angle}(x,y) = 1$$



Transition matrix

$$T_m(x,y) = \begin{cases} \alpha_m & \textbf{if } y = x \\ 1 & \textbf{if } y = x+1 \\ 0 & \textbf{otherwise} \end{cases}$$

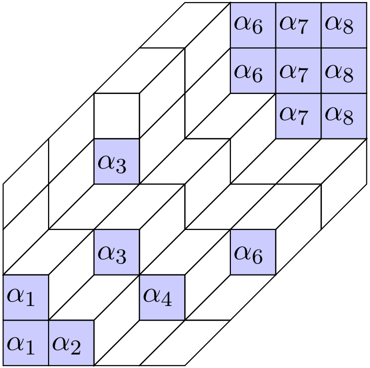
3 Constant weights per column (example)

Weights depend on column

$$w_{\square}(x,y) = \alpha_x,$$

$$w_{\text{yellow}}(x,y) = 1,$$

$$w_{\text{red}}(x,y) = 1$$



Transition matrix

$$T_m(x,y) = \begin{cases} \alpha_m & \textbf{if } y = x \\ 1 & \textbf{if } y = x+1 \\ 0 & \textbf{otherwise} \end{cases}$$

It is Toeplitz matrix with symbol $\varphi_m(z) = z + \alpha_m$

3 Correlation kernel (double contour part only)

Theorem

Correlation kernel at the mth level

$$\frac{1}{(2\pi i)^2}\oint_\gamma\oint_\gamma \left(\prod_{j=m+1}^{L}\varphi_j(w)\right) R_N(w,z) \left(\prod_{j=1}^{m}\varphi_j(z)\right) \frac{w^y}{z^x w^{M+N}} \frac{dz dw}{z}$$

where $R_N(w,z) = \sum_{k=0}^{N-1} \frac{p_k(w)p_k(z)}{h_k}$ **is the reproducing kernel**

for orthogonal polynomials on a contour γ going around 0

$$\frac{1}{2\pi i}\oint_\gamma p_k(z)p_j(z)\frac{\prod_{j=1}^{L}\varphi_j(z)}{z^{M+N}}dz = h_k\delta_{k,j}$$

3 Two periodic parameters

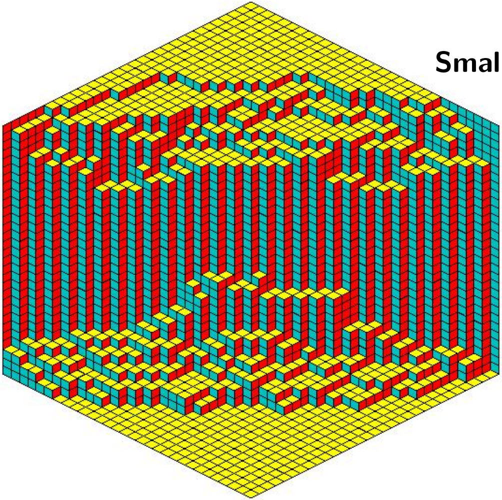
Suppose $$\alpha_m = \begin{cases} 1 & \textbf{if } m \textbf{ is even} \\ \alpha & \textbf{if } m \textbf{ is odd} \end{cases}$$

- **Orthogonality weight is (for $N = M = L - M$)**

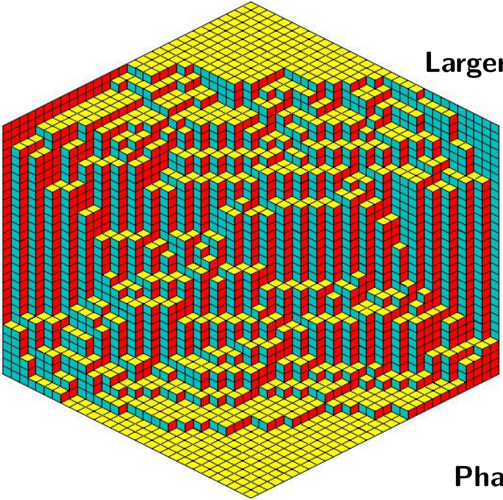
$$\frac{(z+1)^N (z+\alpha)^N}{z^{2N}}$$

- **This model has a phase transition in large N limit**

Charlier, Duits, K, Lenells (2020)



Small parameter $0 < \alpha < 1/9$



Larger parameter $1/9 < \alpha < 1$

Phase transition at $\alpha = 1/9$

- **Asymptotic analysis of the OP with Riemann-Hilbert problem and steepest descent analysis of double integral**

4. Weightings that are periodic in vertical direction

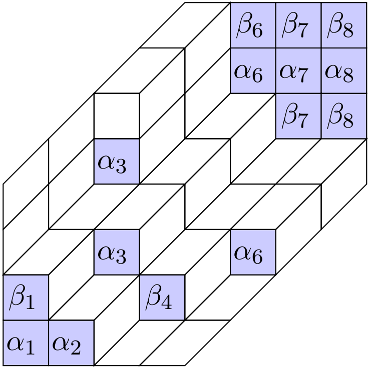
4 Periodicity in vertical direction (example)

$$w_{\square}(x,y) = \begin{cases} \alpha_x, & \textbf{if } y \textbf{ is even} \\ \beta_x, & \textbf{if } y \textbf{ is odd} \end{cases}$$

$$w_{\text{▱}}(x,y) = 1, \quad w_{\text{▱}}(x,y) = 1$$

Transition matrix is block Toeplitz

$$T_m(x,y) = \begin{cases} \alpha_m & \textbf{if } y = x \textbf{ even} \\ \beta_m & \textbf{if } y = x \textbf{ odd} \\ 1 & \textbf{if } y = x+1 \\ 0 & \textbf{otherwise} \end{cases}$$



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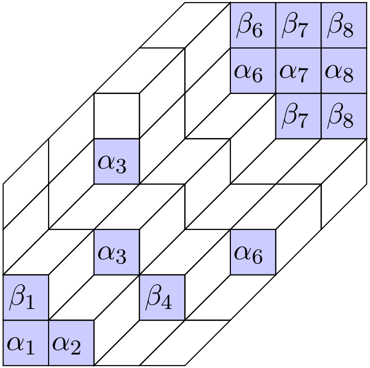

Block symbol

$$\Phi_m(z) = \begin{pmatrix} \alpha_m & 1 \\ z & \beta_m \end{pmatrix}$$

4 Correlation kernel (double contour part only)

Theorem

Correlation kernel at the mth level (for $2N \times 2M \times L - 2M$ hexagon) are entries of

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma \left(\prod_{j=m+1}^{L} \Phi_j(w) \right) R_N(w,z) \left(\prod_{j=1}^{m} \Phi_j(z) \right) \frac{w^y}{z^x w^{M+N}} \frac{dz dw}{z}$$

where $R_N(w,z) = \sum_{k=0}^{N-1} P_k^T(w) H_k^{-1} P_k(z)$ **is the reproducing kernel for matrix valued orthogonal polynomials on a contour γ going around 0**

$$\frac{1}{2\pi i} \oint_\gamma P_k(z) \frac{\prod_{j=1}^{L} \Phi_j(z)}{z^{M+N}} P_j(z)^T dz = H_k \delta_{k,j}$$

4 Comment

- **The theorem extends to transition matrices with block Toeplitz structure of any periodicity.**

5. Matrix valued orthogonal polynomials (MVOP)

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$$\frac{1}{2\pi i}\oint_\gamma P_k(z)W(z)P_j^T(z)dx = H_j\delta_{j,k}, \qquad \det H_j \neq 0$$

- $W(z)$ **is** $p \times p$ **matrix for every** z
- P_k **is matrix valued polynomial**

$$P_k(x) = C_0x^k + C_1x^{k-1} + \cdots, \qquad C_i \text{ is } p \times p \text{ matrix.}$$

- **Integral is taken entry-wise.**

5 Matrix valued orthogonal polynomials (MVOP)

$$\frac{1}{2\pi i}\oint_{\gamma} P_k(z)W(z)P_j^T(z)dx = H_j\delta_{j,k}, \qquad \det H_j \neq 0$$

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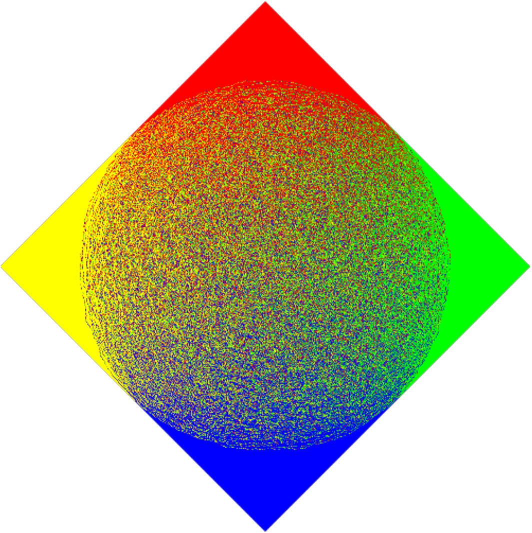
- **Integral is taken entry-wise.**

Questions on existence and uniqueness, recurrence relations, generating functions, differential equations, ...

- **Examples and Applications: do MVOP appear in "real life" problems?**

6. Two periodic Aztec diamond

6 Random tiling with uniform measure

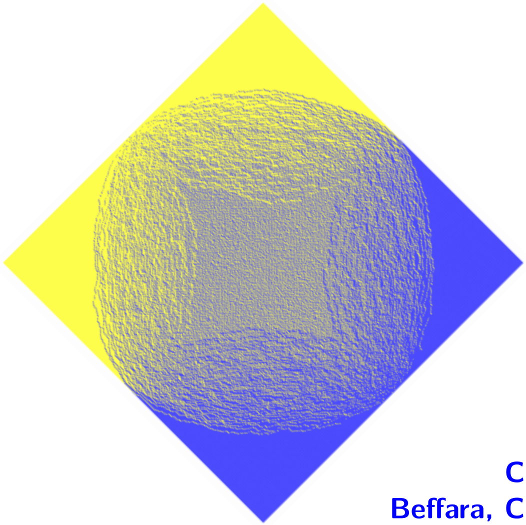


Deterministic pattern near corners
Solid region
or frozen region

Disorder in the middle
Liquid region

Boundary curve
Arctic circle

6 Aztec diamond; two-periodic weighting

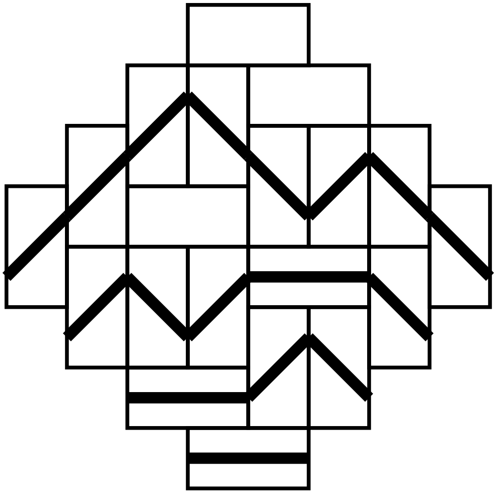


A new phase within the liquid region:

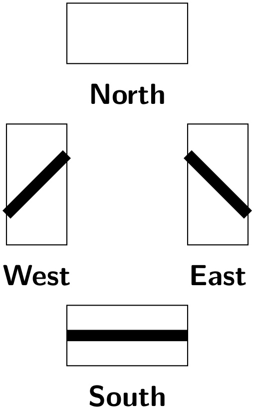
gas region
(smooth region)

Chhita, Johansson (2016)
Beffara, Chhita, Johansson (2018)

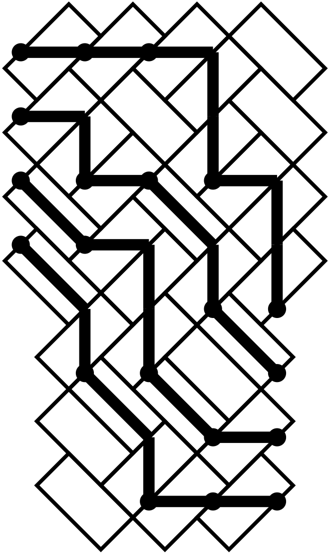
6 Paths in the Aztec diamond



Line segments on West, East and South dominos



6 Transformations and extension; particle system



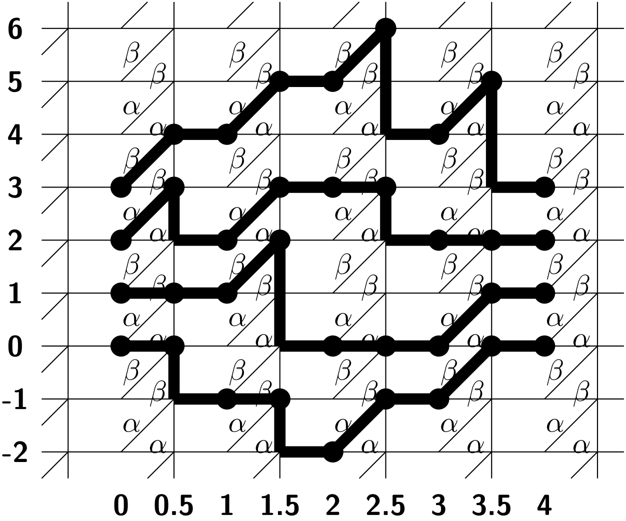
- Rotate the Aztec diamond
- Extend the tiling to a **double Aztec diamond**
- Put particles on the paths
- Particles are a **determinantal point process**

6 Non-intersecting paths on a weighted graph

- **Apply affine transformation**

Two periodic weighting

- **Bernoulli step with weight α or $\beta = \alpha^{-1}$**
- **Steps down plus horizontal step have weight 1**



6 Symbols and weight

Block symbols are $\begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}$ **and** $\frac{1}{z-1}\begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$

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Block symbols are $\begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}$ and $\frac{1}{z-1}\begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$

Weight matrix is W^N for Aztec diamond of size $2N$, where

$$W(z) = \frac{1}{z(z-1)^2}\begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}\begin{pmatrix} z & 1 \\ z & z \end{pmatrix}\begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}\begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$$

$$= \frac{1}{(z-1)^2}\begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

6 MVOP

MVOP of degree N is explicit if N is even

$$P_N(z) = (z-1)^N W(\infty)^{N/2} W^{-N/2}(z)$$

- **The double contour integral for the correlation kernel simplifies considerably**
- **Different approach is due to Berggren-Duits (2019)**

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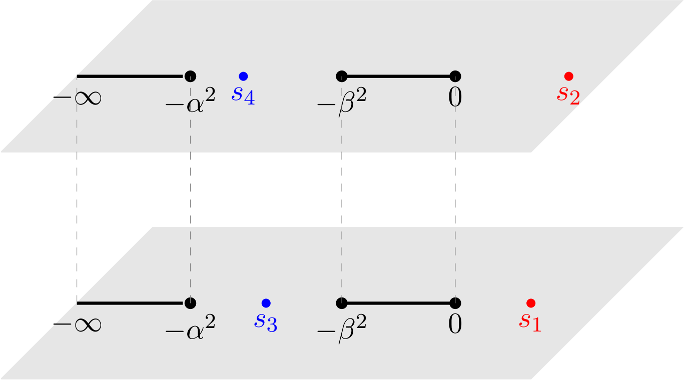
- **The double contour integral for the correlation kernel simplifies considerably**
- **Different approach is due to Berggren-Duits (2019)**

- **What remains is saddle point analysis of the double contour integral.**
- **There are four saddle points (depending on position in the Aztec diamond) that "live" on two-sheeted spectral curve**

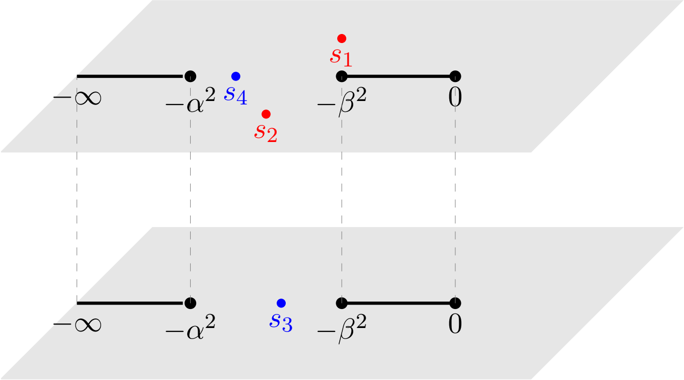
$$y^2 = z\left(z+\alpha^2\right)\left(z+\beta^2\right)$$

6 Solid phase



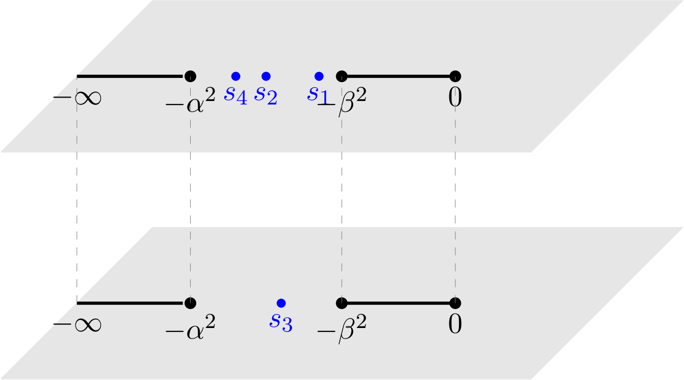
- **At least two saddles are in** $[-\alpha^2, -\beta^2]$
- **Other saddles** s_1 **and** s_2 **are in** $[0, \infty)$ $\implies$ **solid phase**

6 Liquid phase



- **Saddles s_1 and s_2 are not real** $\implies$ **liquid phase**

6 Gas phase



- **All saddles are in $[-\alpha^2, -\beta^2]$ $\implies$ Gas phase**

6 Phase diagram

Thank you for your attention