Universidad Zaragoza 1542

Instituto Universitario de Investigación de Matemáticas y Aplicaciones Universidad Zaragoza

ECM 2020 PORTOROŽ

KHRUSHCHEV'S FORMULAS FOR ORTHOGONAL POLYNOMIALS

Joint works with
M.J. Cantero, L. Moral, U Zaragoza
F.A. Grünbaum, UC Berkeley
C. Cedzich, R.F. Werner, U Hannover
A.H. Werner, U Copenhagen

A motivation

A MOTIVATION

μ **PROBABILITY MEASURE** on $S \subset \mathbb{C}$ ➡ (p_n) **ORTHONORMAL POLYNOMIALS**

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The talk deals with a **ready-made tool** to tackle this kind of problems when $S = \mathbb{T}, \mathbb{R}$ which is valid also for **unbounded** measures even in the **indeterminate** case

Orthogonal Polynomials (OP)

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μ **measure** on
the **Real Line**

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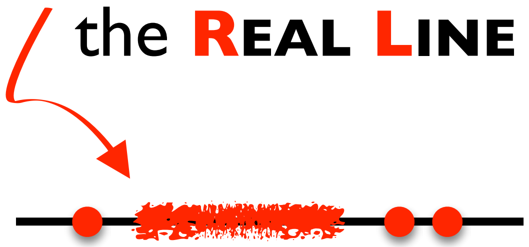

OPRL

(p_n)

RR

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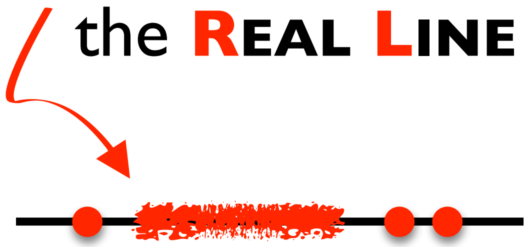

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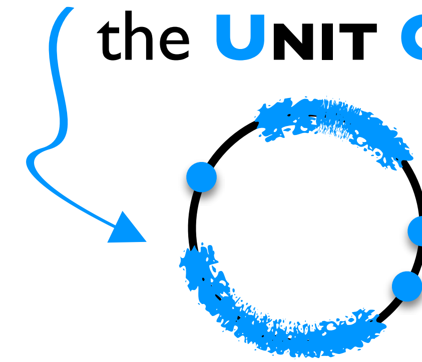

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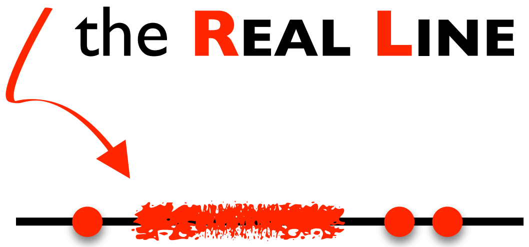

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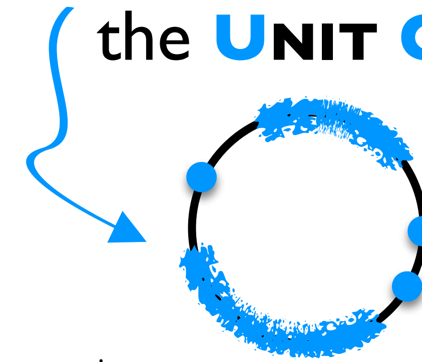

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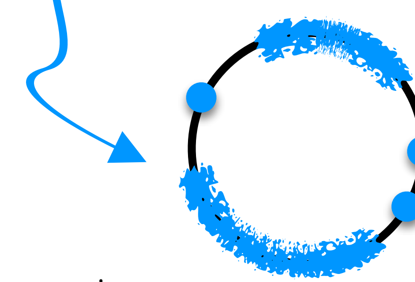

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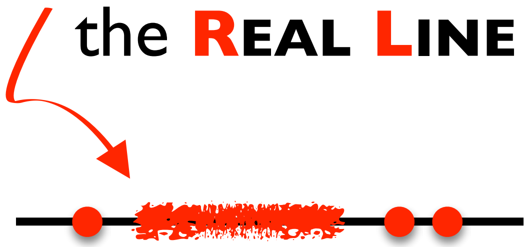

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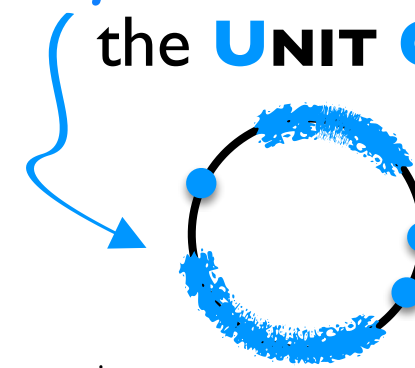

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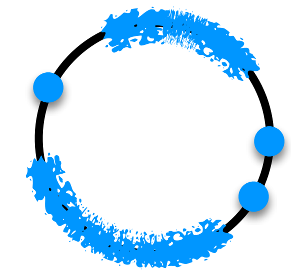

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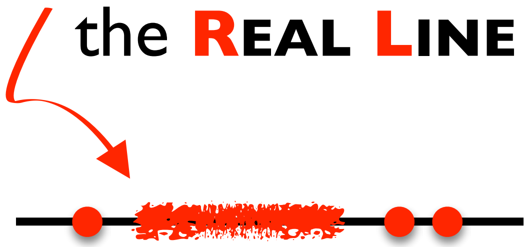

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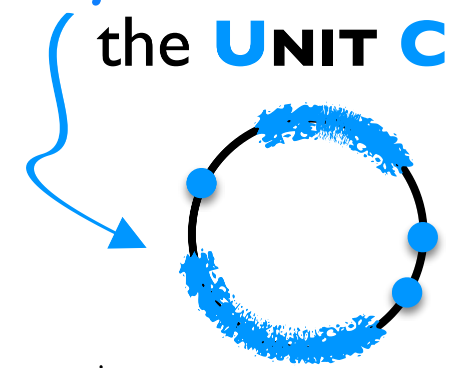

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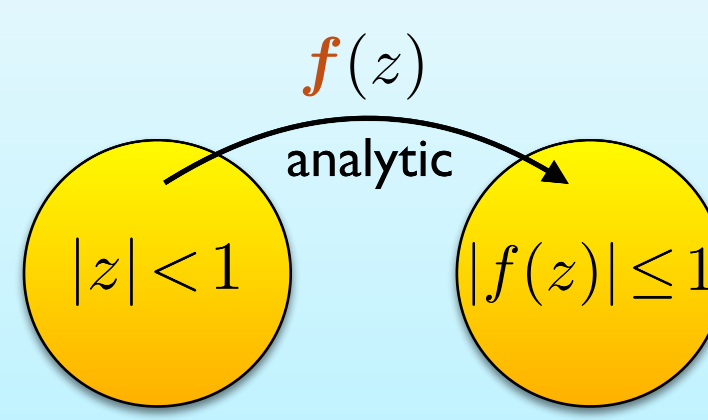

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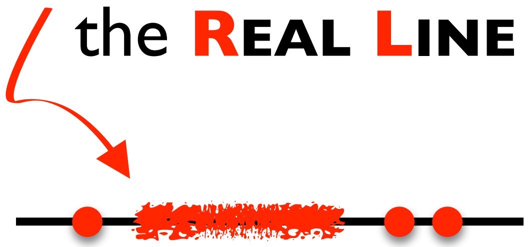

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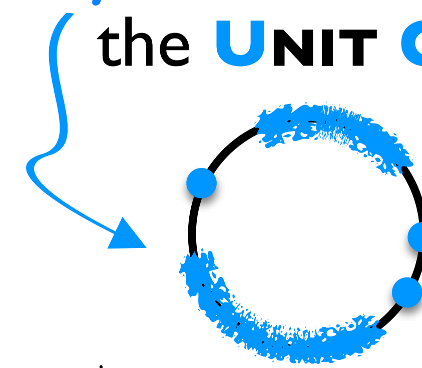

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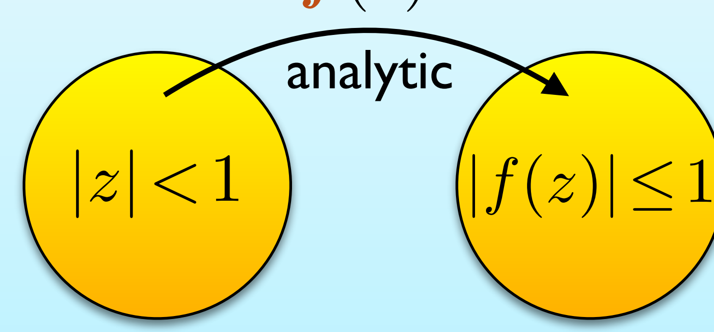

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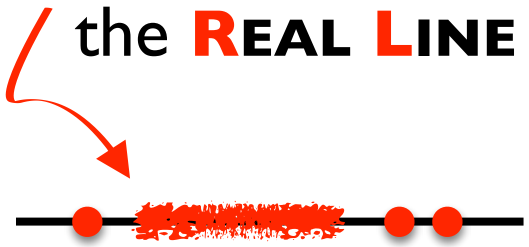

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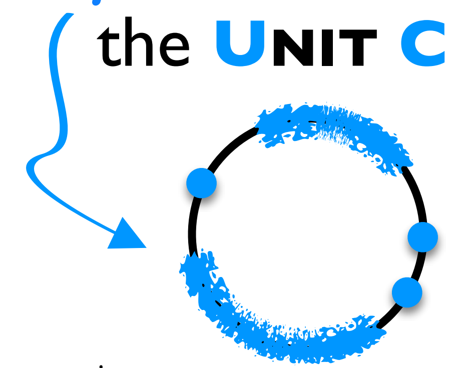

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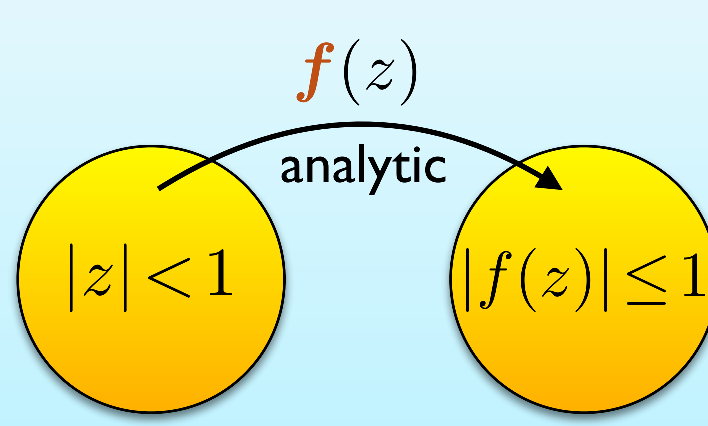

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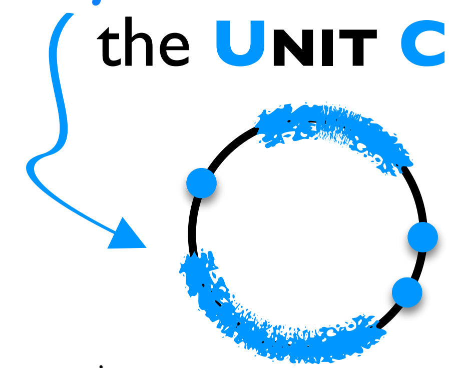

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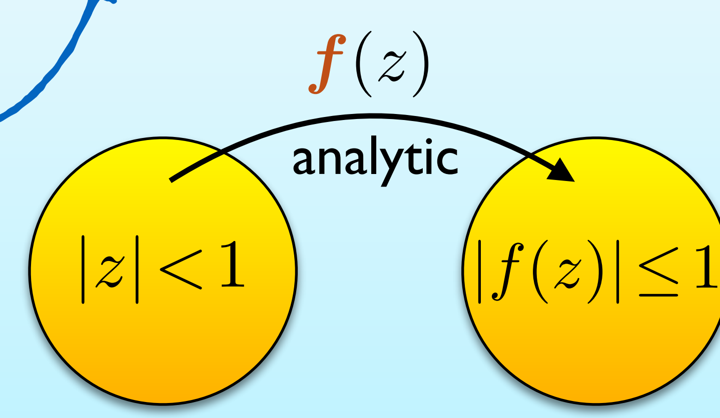

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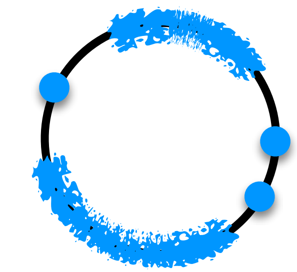

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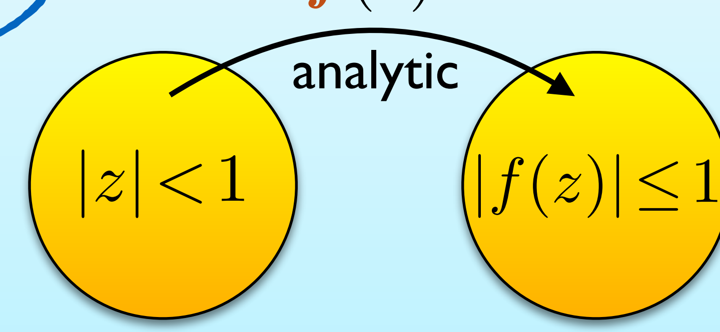

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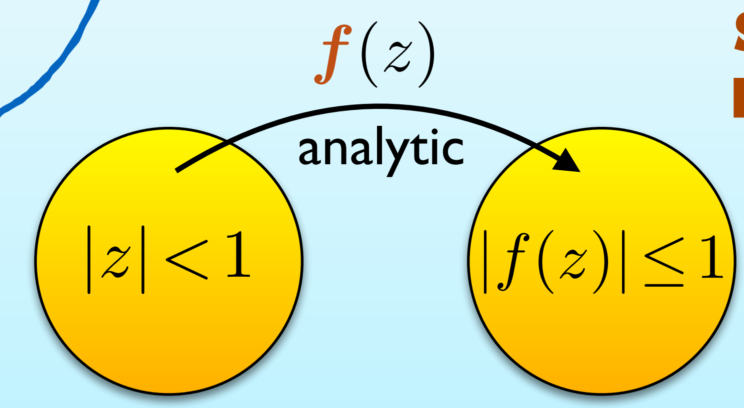

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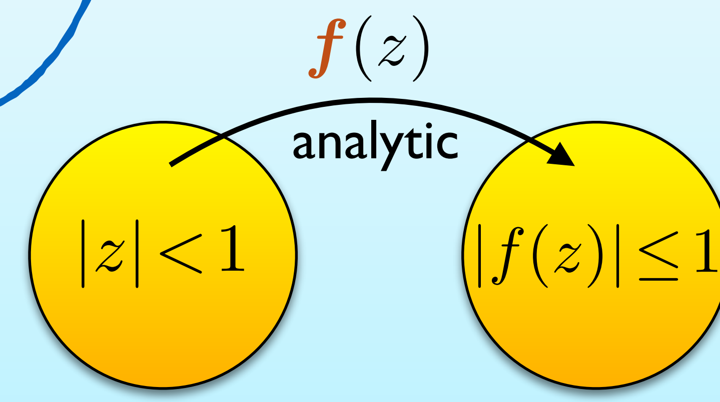

$$f \xleftrightarrow{\text{one-to-one}} (\alpha_0, \alpha_1, \alpha_2, \dots)$$

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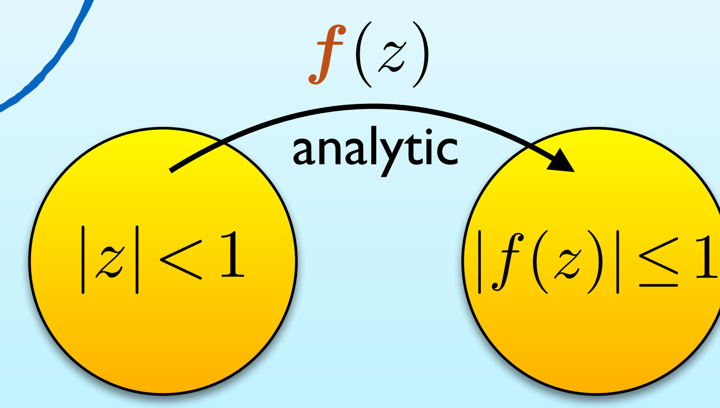

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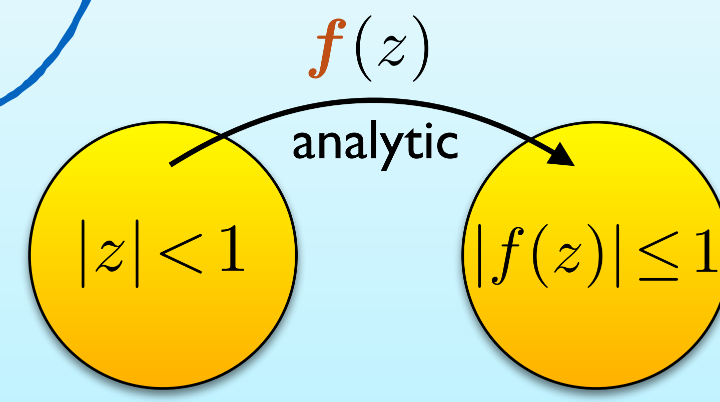

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$$f \equiv (\alpha_0, \alpha_1, \alpha_2, \dots) \qquad f_n \equiv (\alpha_n, \alpha_{n+1}, \dots)$$

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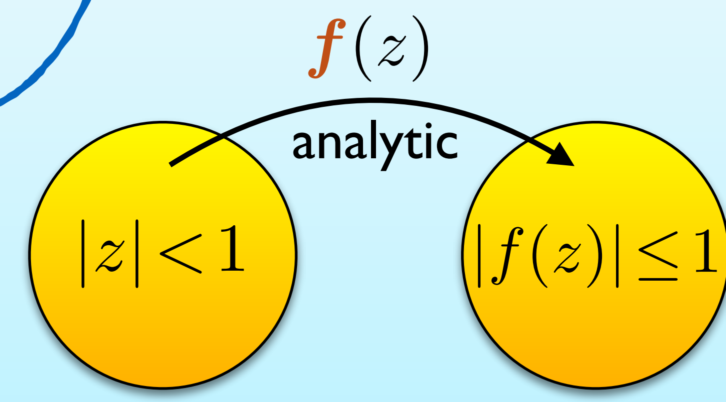
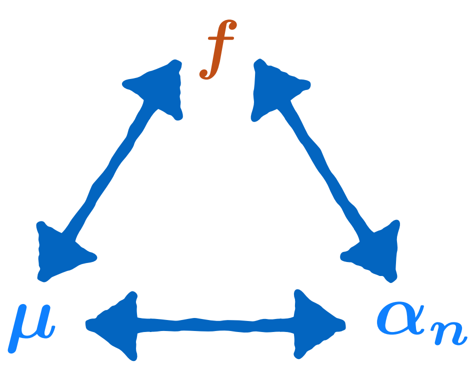


$$f \equiv (\alpha_0, \alpha_1, \alpha_2, \dots) \qquad f_n \equiv (\alpha_n, \alpha_{n+1}, \dots)$$



OPUC & Schur functions

OPUC shortcut

$\mu \longleftrightarrow f \longleftrightarrow \alpha_n$

Schur fctions

$$\int_{\mathbb{T}} \frac{d\mu(t)}{1-zt} = \frac{1}{1-zf(z)}$$

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$$f_0(z) = f(z) \qquad f_{n+1}(z) = \frac{1}{z}\,\frac{f_n(z)-f_n(0)}{1-\overline{f_n(0)}f_n(z)}$$

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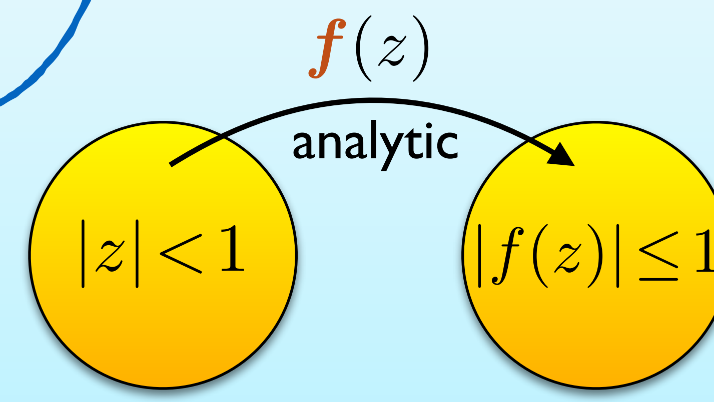

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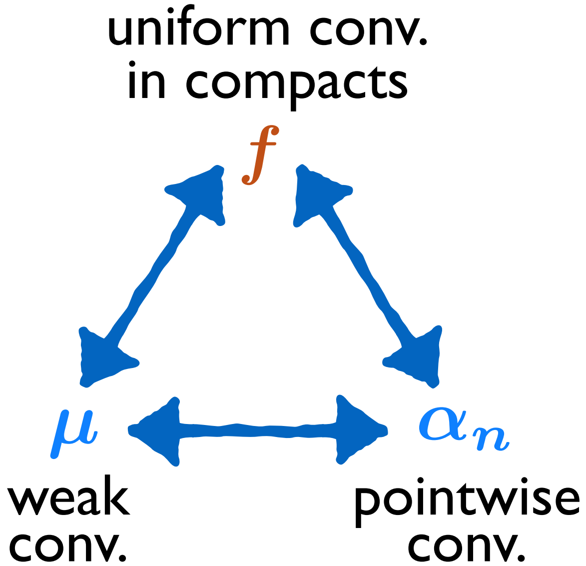

Homeomorphisms

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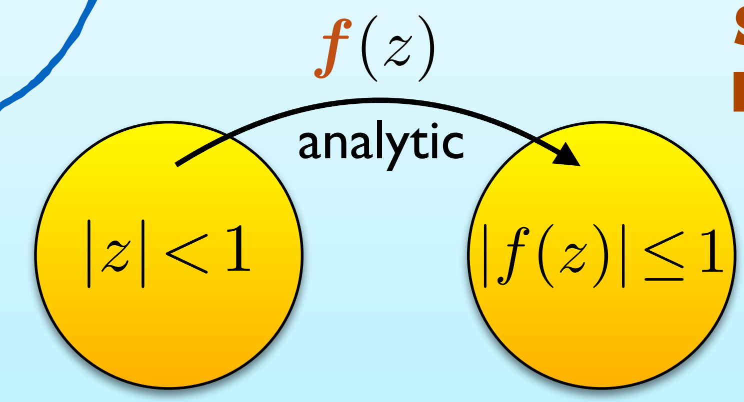

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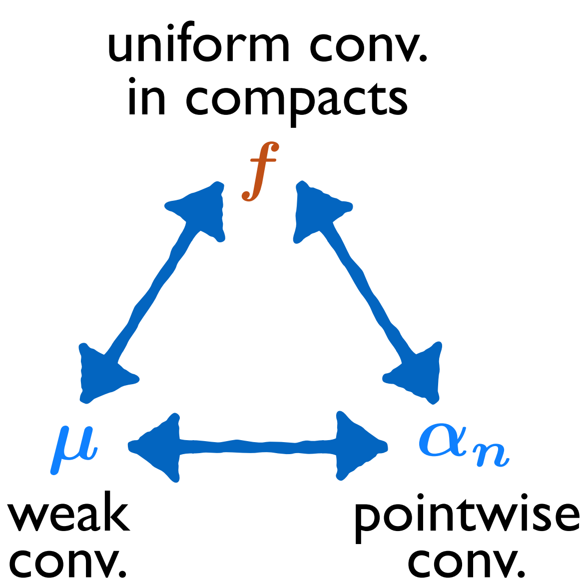

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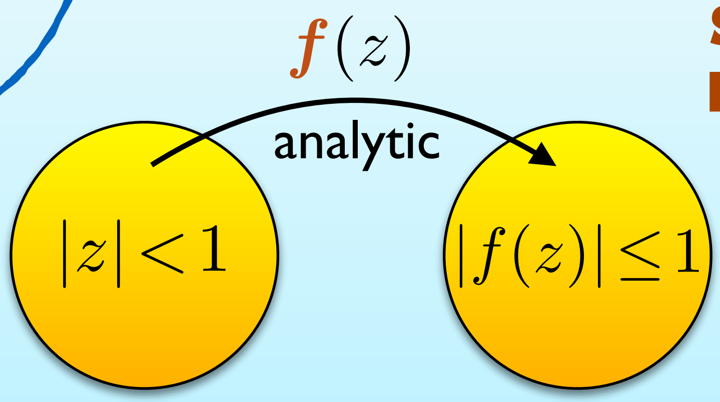

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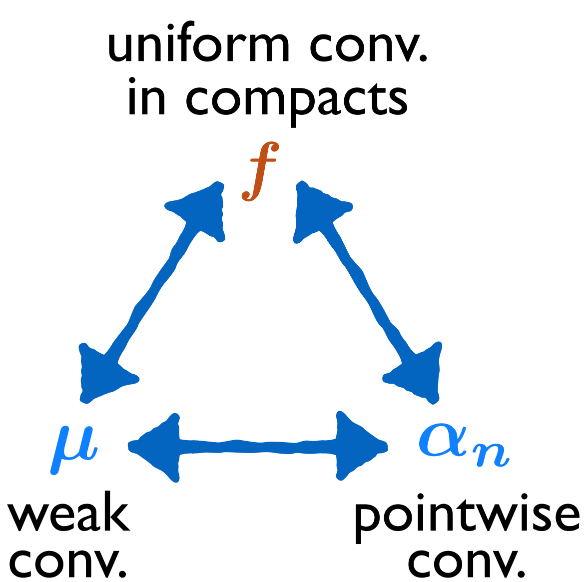

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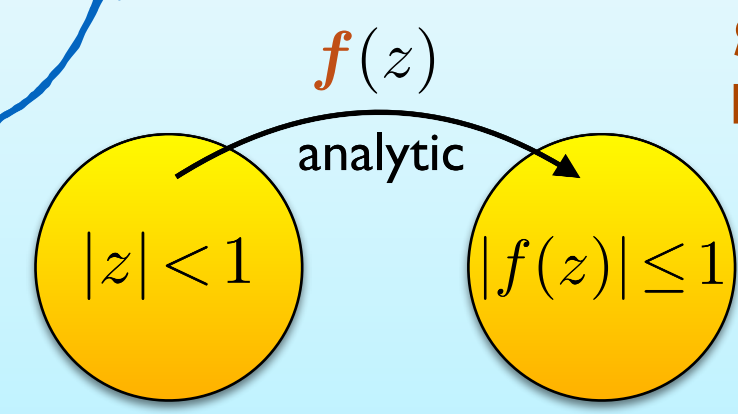

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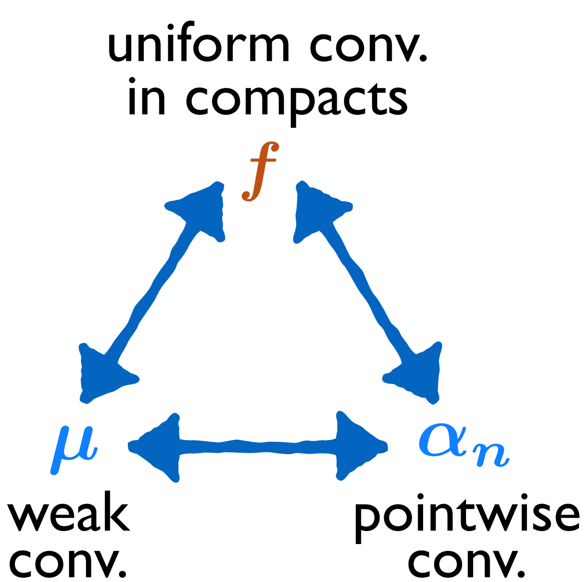

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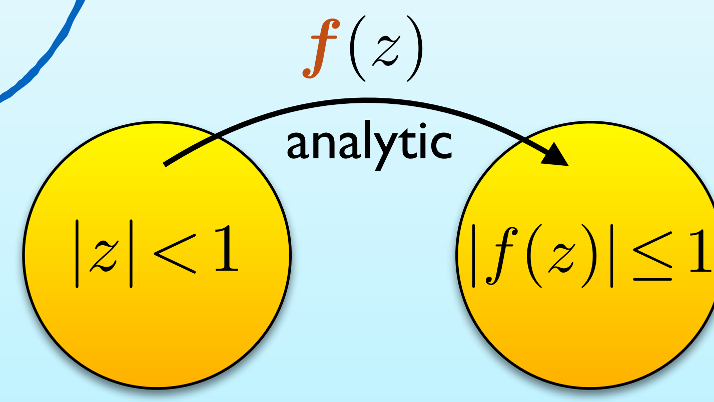

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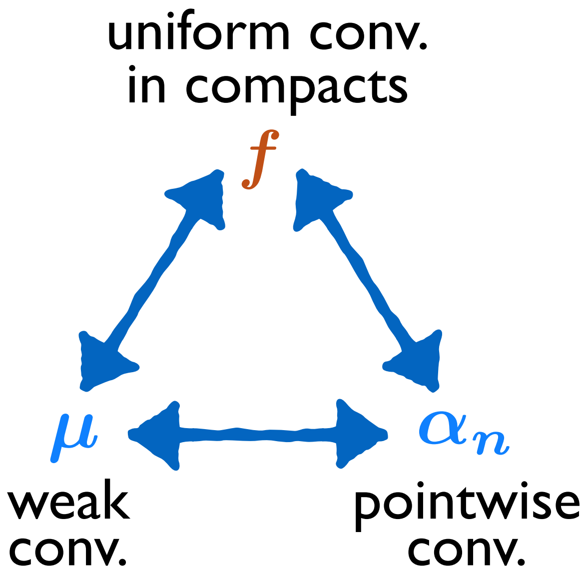

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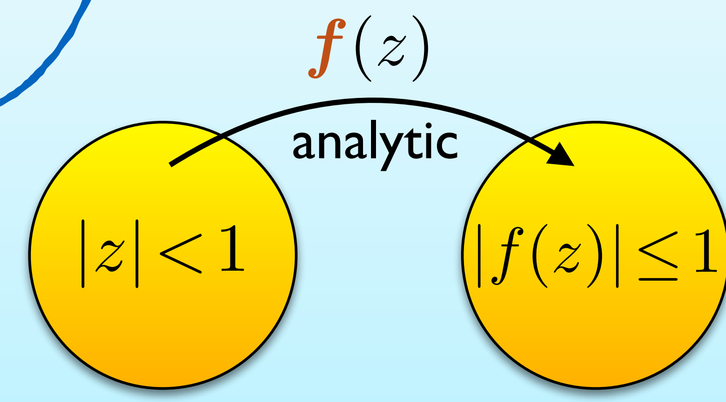

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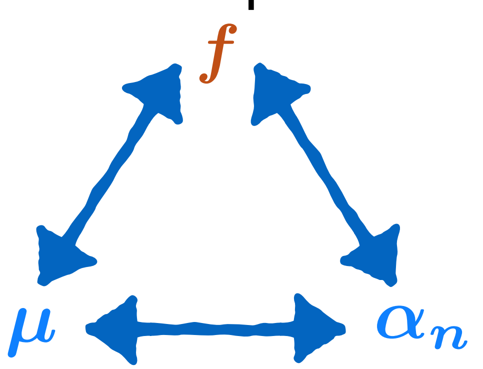
uniform conv. in compacts



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The problem posed at the beginning!

What is the Schur fction $f^{[n]}$ of $|\varphi_n|^2 d\mu$?

OPUC & Schur functions

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OPUC & SCHUR FUNCTIONS

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INVERSE ITERATES

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ITERATES (f_n)

INVERSE ITERATES (g_n)

KHRUSHCHEV'S FORMULA

Based on this result, the beginning of XXIth century sees and **OPUC revolution** due to **S. Khrushchev**, including the solution to the initial problem for OPUC

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$|\varphi_n|^2 d\mu$ conv. $\Leftrightarrow$ $f_n g_n$ conv. $\Leftrightarrow$ $\alpha_n \alpha_{n+\ell} \to 0 \;\; \forall \ell \in \mathbb{N}$ or there exist

$a, a' > 0, \;\; \lambda \in \mathbb{T}, \;\; k \in \mathbb{N}, \;\; \ell \in \{0, \dots, k-1\}$ s.t.

$$|\alpha_{2nk+\ell+j}| \to \begin{cases} a & \text{if } j = 0 \\ a' & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

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Schur techniques for OPRL?

“Schur” for OPRL

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Main idea: Defining **OPRL analogue of Schur fctions** via

"SCHUR" FOR OPRL

MAIN IDEA: Defining **OPRL analogue of SCHUR FCTIONS** via

MEASURE μ → $$\int \frac{d\mu(t)}{1-zt} = \frac{1}{1-zf(z)}$$ → **"SCHUR" FCTION** f

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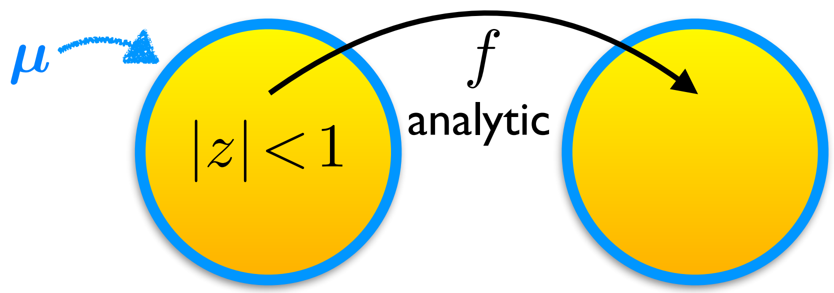
$\operatorname{supp}\mu \subset \mathbb{T} \;\Rightarrow\; f(z)$ **Schur** fction



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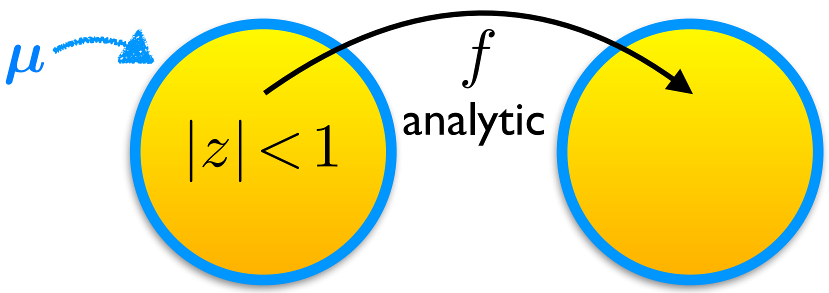
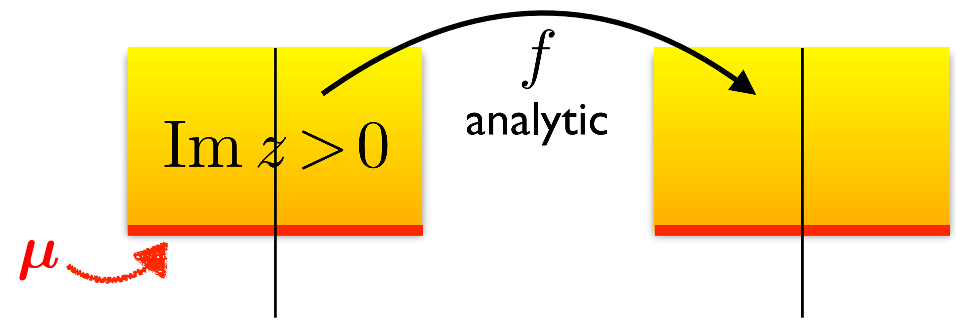


supp $\mu \subset \mathbb{R} \;\Rightarrow\; f(z)$ **Nevanlinna** fction



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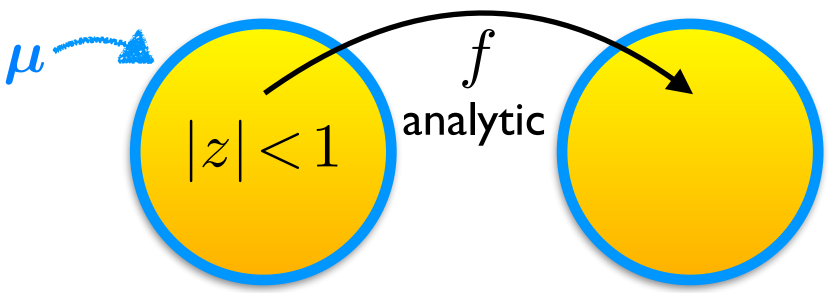

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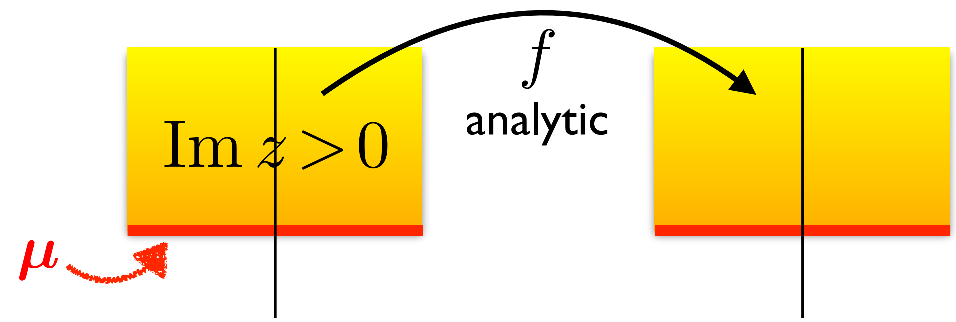

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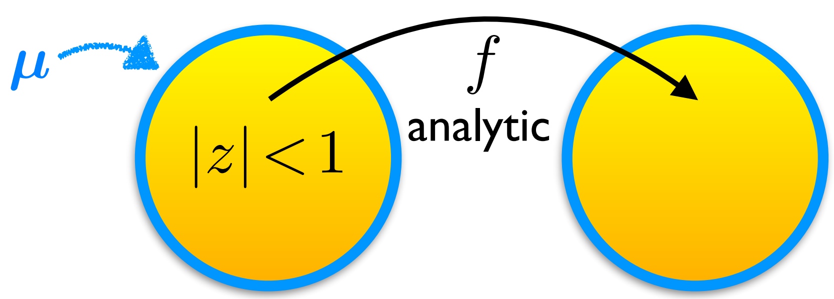

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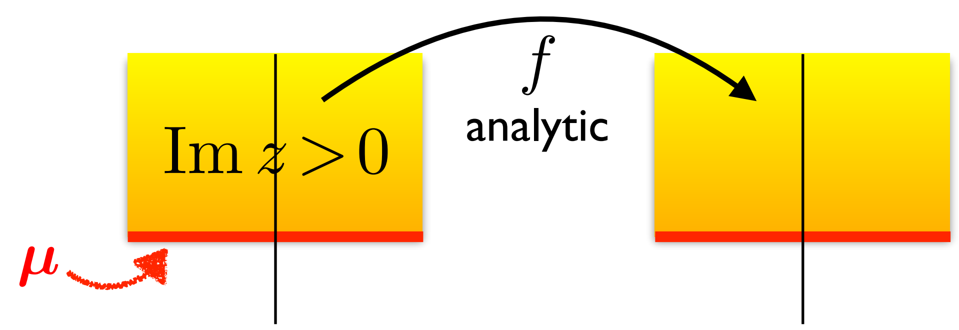

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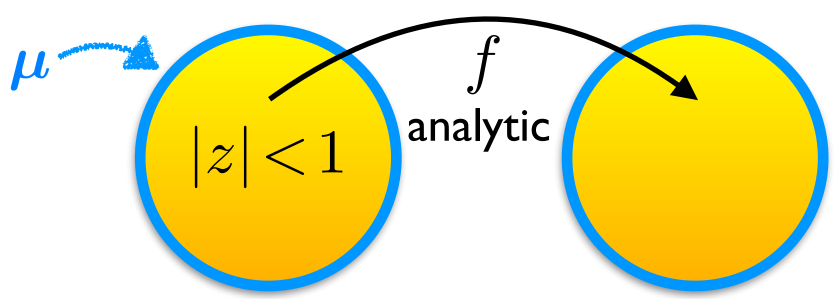
supp $\mu \subset \mathbb{T} \;\Rightarrow\; f(z)$ **Schur** fction



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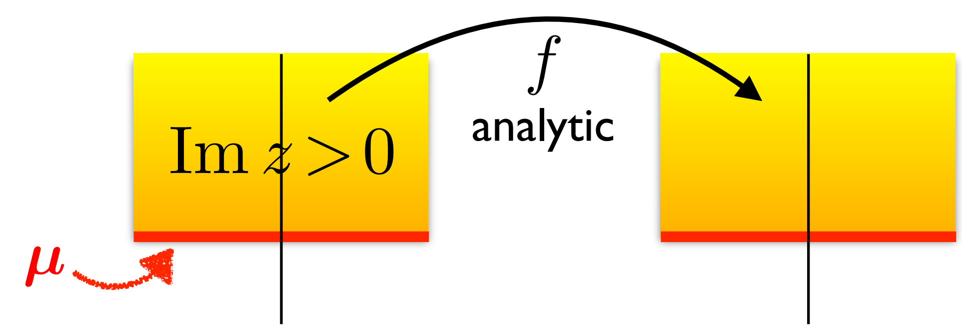

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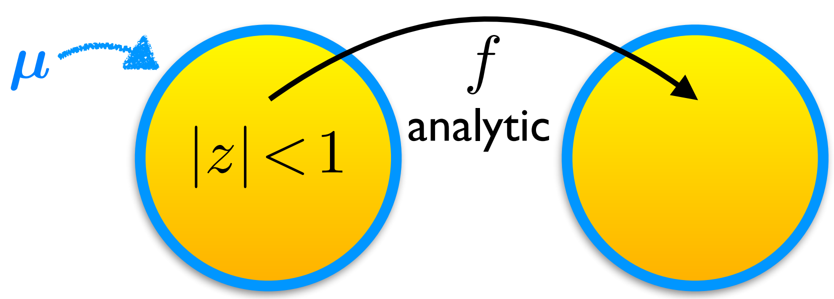

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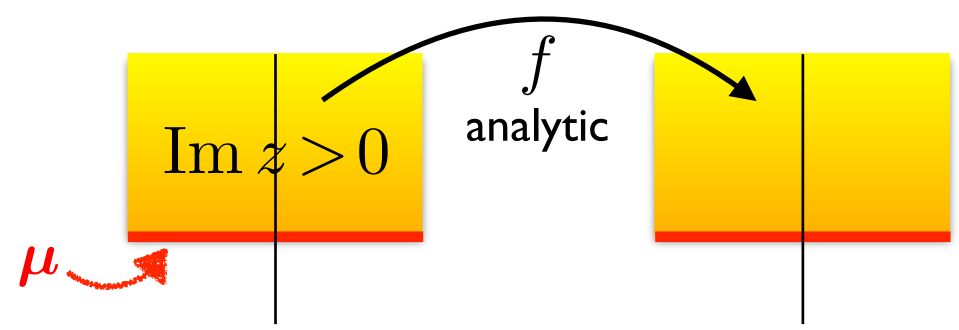

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OPRL Khrushchev's formula

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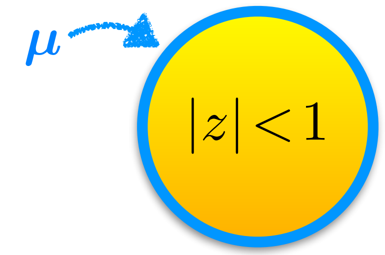
OPUC **(known)**

RR parameters α_n



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OPUC
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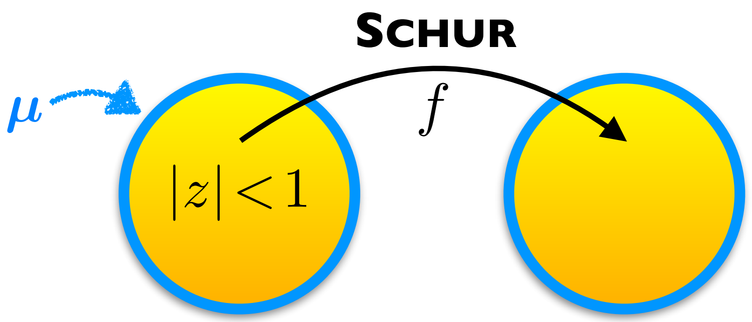

OPRL KHRUSHCHEV'S FORMULA

OPUC (known)

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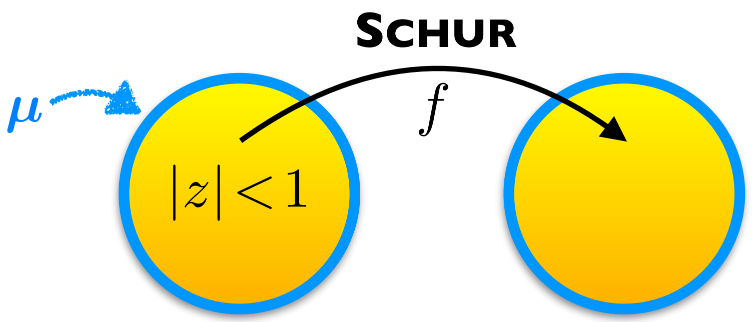
SCHUR parameters



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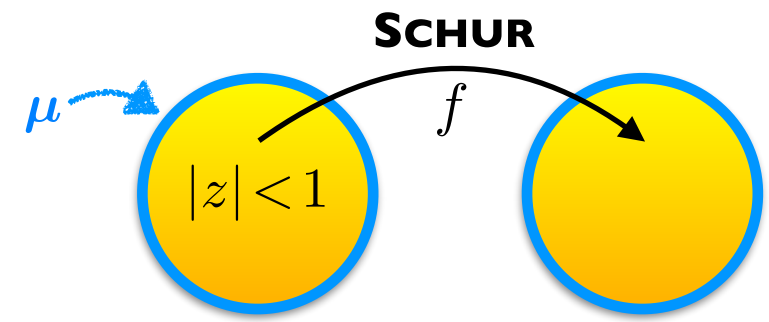

Khrushchev factorization

Schur fction of $|\varphi_n|^2 d\mu = f_n\, g_n$

$$\begin{cases} f_n \equiv (\alpha_n, \alpha_{n+1}, \dots) & \text{ITERATES} \\ g_n \equiv (-\overline{\alpha}_{n-1}, \dots, -\overline{\alpha}_0, 1) & \text{INVERSE ITERATES} \end{cases}$$

OPRL Khrushchev's formula

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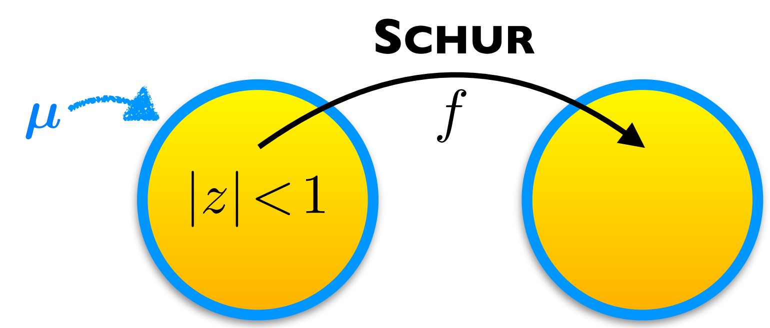
SCHUR parameters

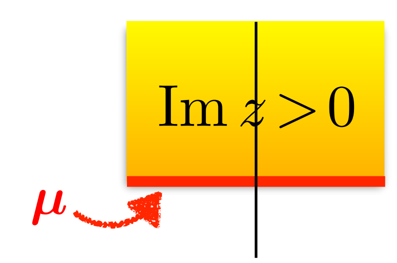


Khrushchev FACTORIZATION

Schur fction of $|\varphi_n|^2 d\mu = f_n\, g_n$

$$\begin{cases} f_n \equiv (\alpha_n, \alpha_{n+1}, \dots) & \textbf{ITERATES} \\ g_n \equiv (-\overline{\alpha}_{n-1}, \dots, -\overline{\alpha}_0, 1) & \textbf{INVERSE ITERATES} \end{cases}$$

OPRL (**new!**) **RR** parameters $a_n,\ b_n$



OPRL KHRUSHCHEV'S FORMULA

OPUC
(known)

RR parameters α_n

$$\int_{\mathbb{T}} \frac{d\mu(t)}{1-zt} = \frac{1}{1-zf(z)} \quad \longrightarrow \quad f \equiv (\alpha_0, \alpha_1, \dots)$$

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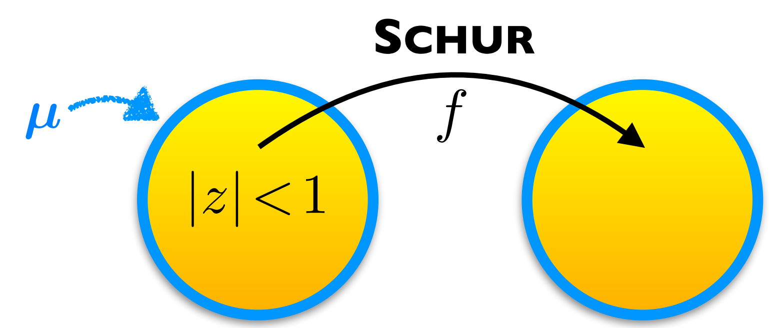

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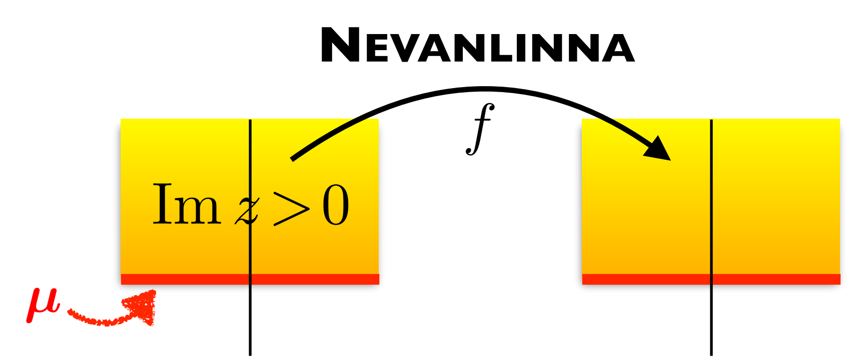

OPRL Khrushchev's formula

OPUC
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RR parameters α_n

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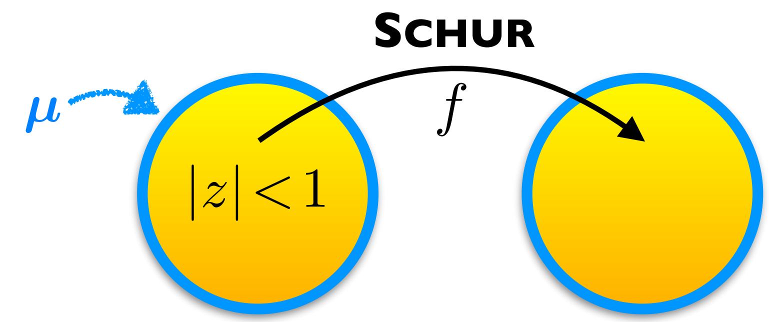

Khrushchev factorization

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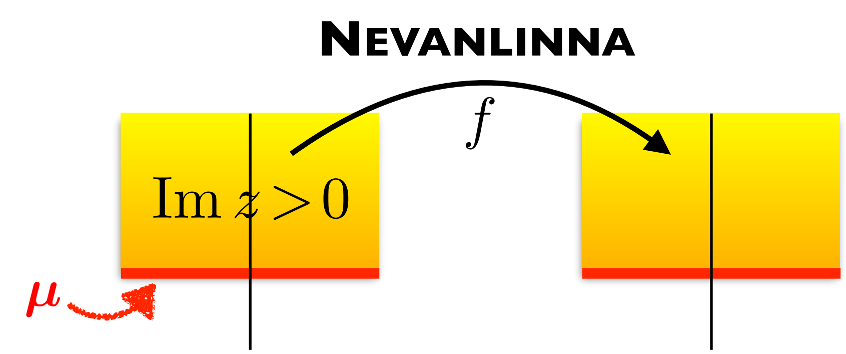
"SCHUR" parameters



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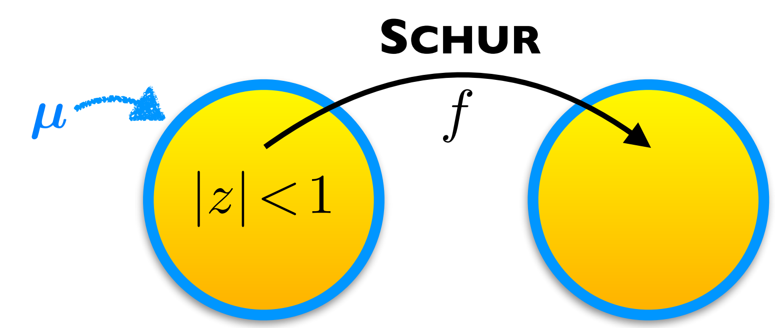

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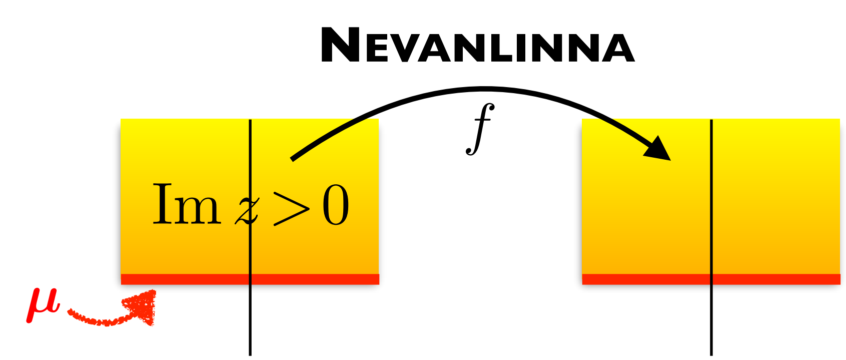

KHRUSHCHEV DECOMPOSITION

Nevanlinna fction of $|p_n|^2\, d\mu = f_n + g_n$

$$\begin{cases} f_n \equiv (b_n, a_n, b_{n+1}, a_{n+1}, \ldots) & \text{ITERATES} \\ g_n \equiv (0, a_{n-1}, b_{n-1}, \ldots, a_0, b_0) & \text{INVERSE ITERATES} \end{cases}$$

OPUC-OPRL DIFFERENCES

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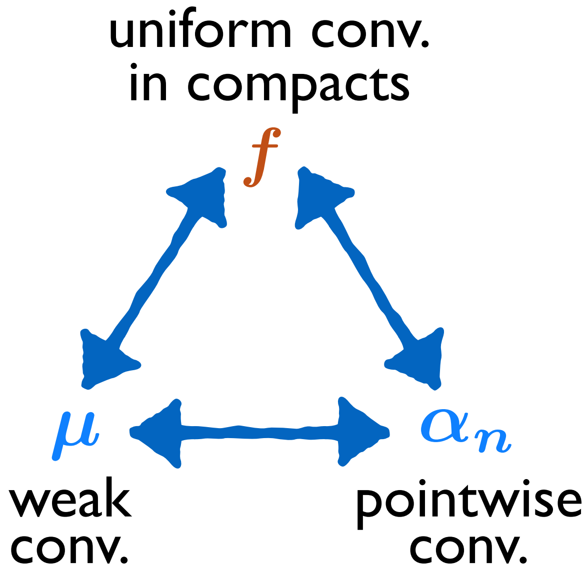

Homeomorphisms

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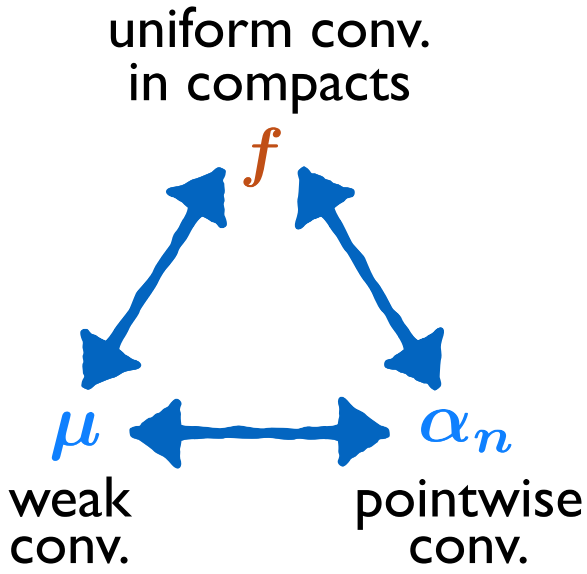

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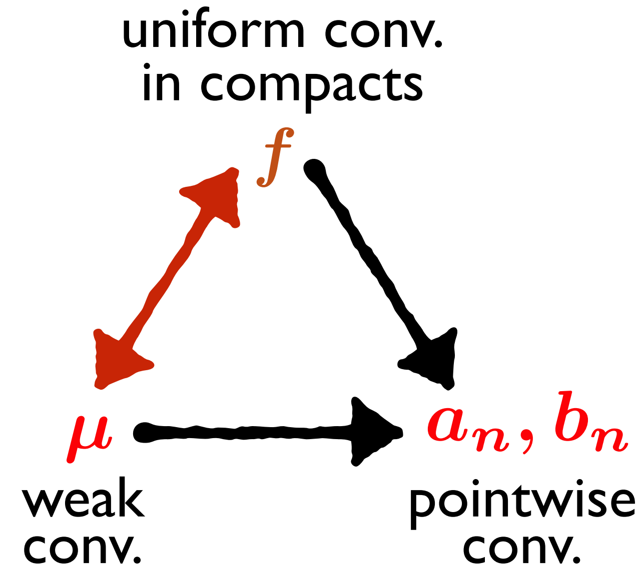

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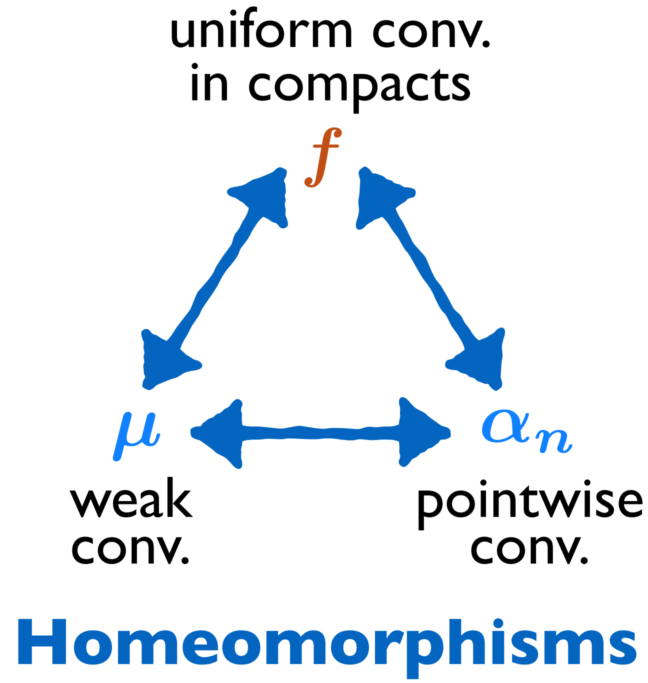

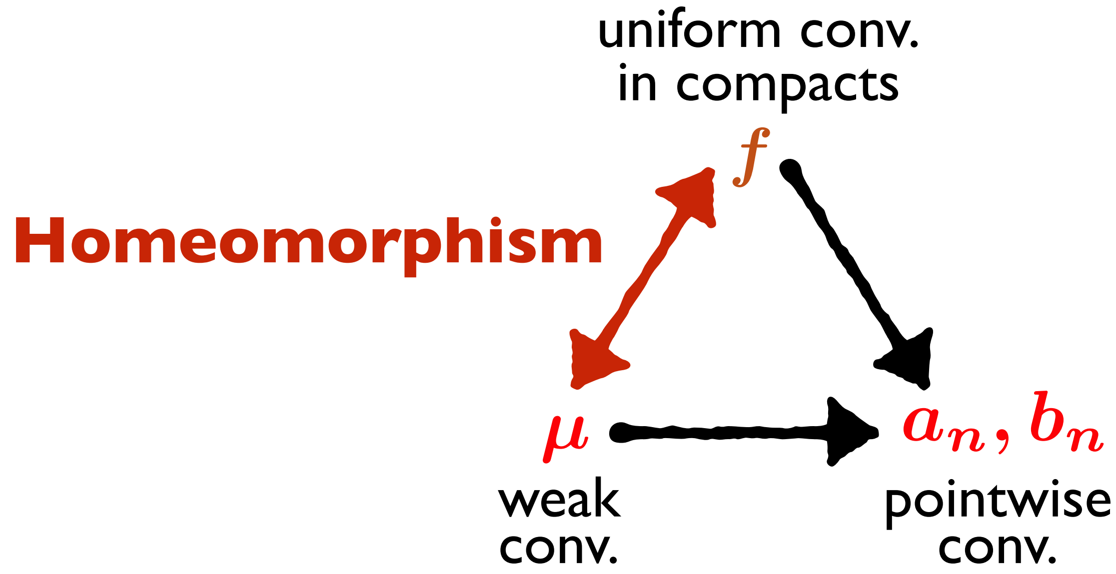

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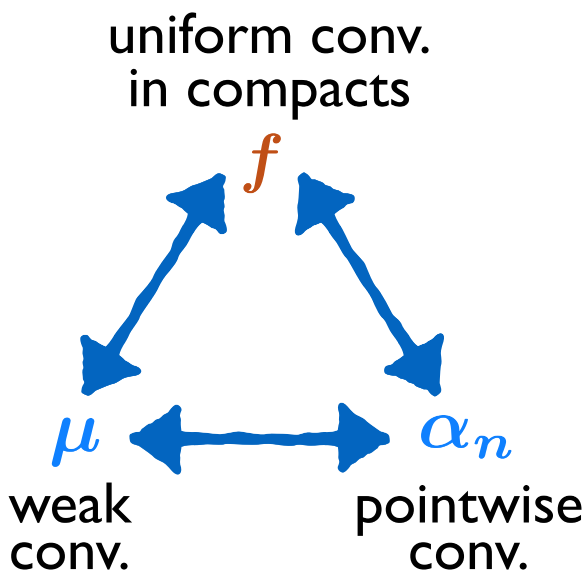

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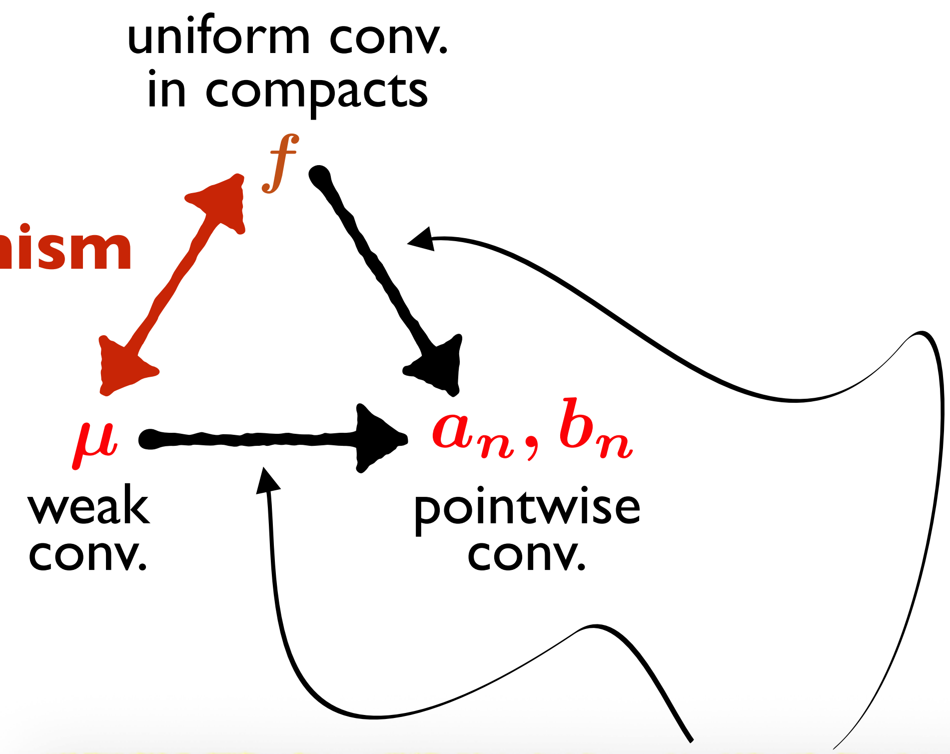
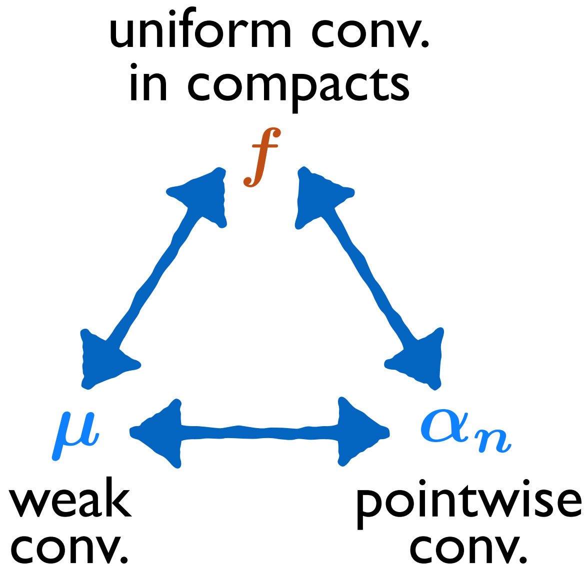


Fail to be one-to-one and continuous
in the unbounded case

OPUC-OPRL DIFFERENCES



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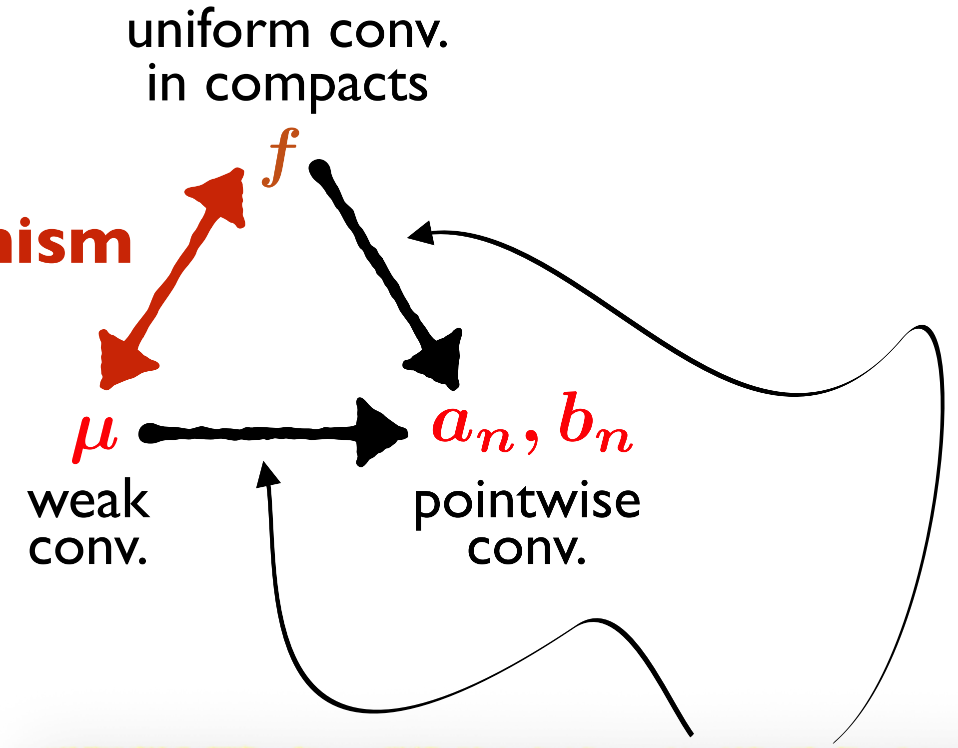

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OPRL applications?

OPRL APPLICATIONS

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$|p_n|^2 d\mu$ conv.

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$$\Leftarrow \quad \begin{array}{l} a_{2n} \to \infty, \quad a_{2n+1} \to 0 \\ b_{2n} + a_{2n}, \quad b_{2n+1} + a_{2n} \ \text{conv.} \end{array}$$

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More precisely, if $b_{2n} + a_{2n} \to c, \ b_{2n+1} + a_{2n} \to c'$ then $|p_n|^2 d\mu \to \frac{1}{2}\delta_{(c+c')/2}$

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OPRL Krhushchev's formula is also an effective tool to study the whole set of **limit points** of $|p_n|^2 d\mu$

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OPRL Krhushchev's formula is also an effective tool to study the whole set of **limit points** of $|p_n|^2 d\mu$

A **matrix valued version** exists, which helps to the convergence analysis of $p_n^\dagger d\mu\, p_n$ for matrix OPRL

Schur, Nevanlinna & operators

Our definition of **"Schur" fctions** is linked to
operator representations as **"first return" fctions**

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$$f(z) = \langle \psi | T(1 - zQT)^{-1} \psi \rangle$$

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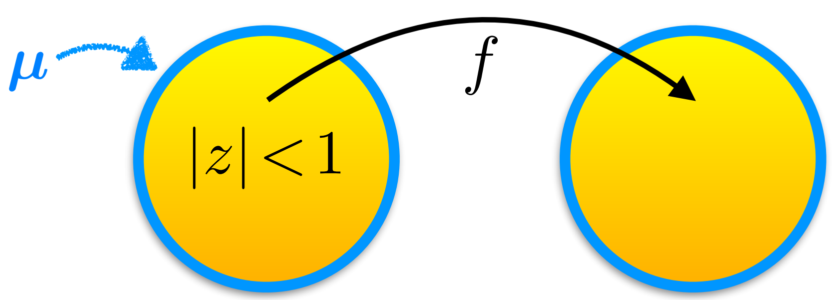

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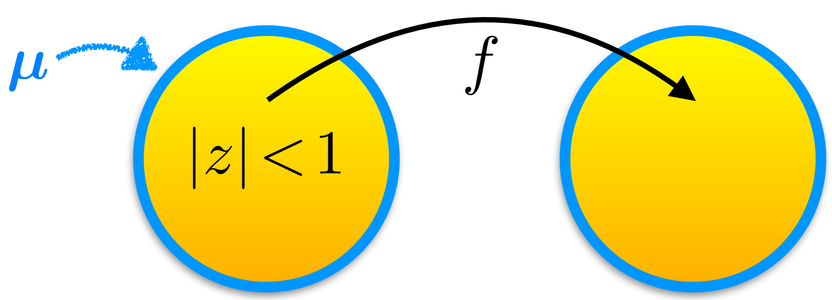
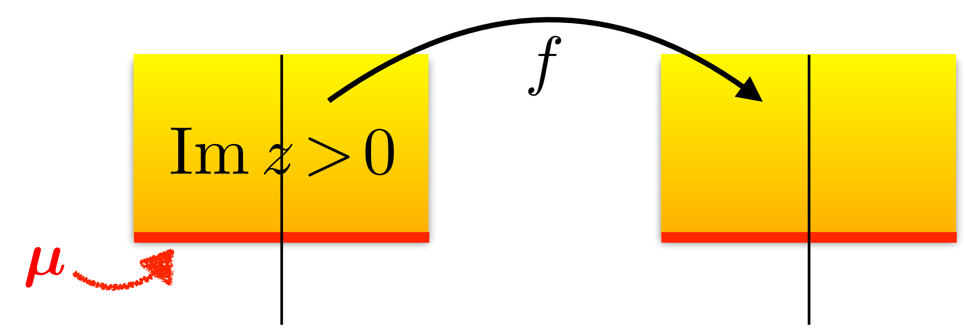


T **self-adjoint** $\Rightarrow f(z)$ **Nevanlinna** fction



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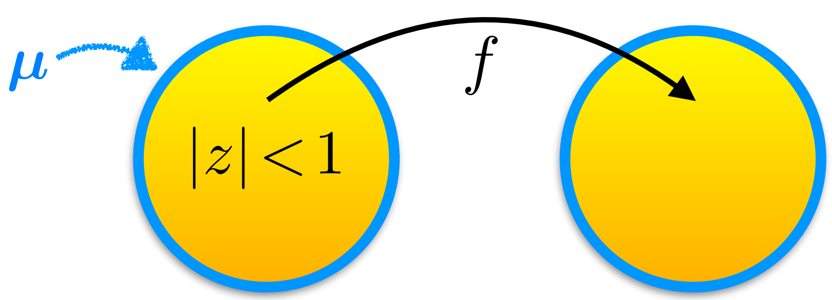

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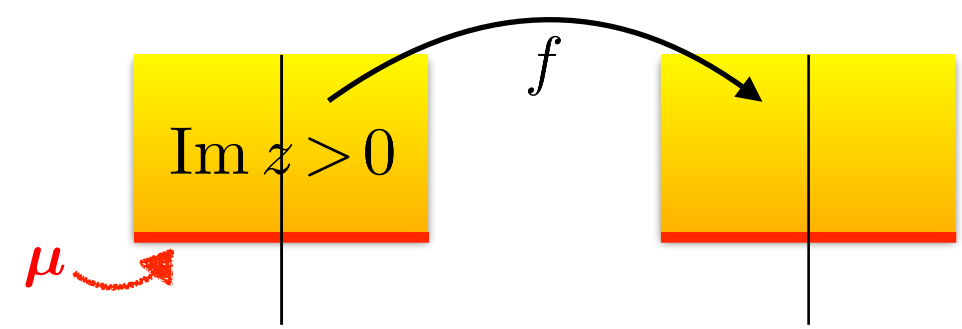

This yields **NEW operator representations** of Schur & Nevanlinna fctions

General Khrushchev's Formulas

Khrushchev's formulas **hold for arbitrary operators**

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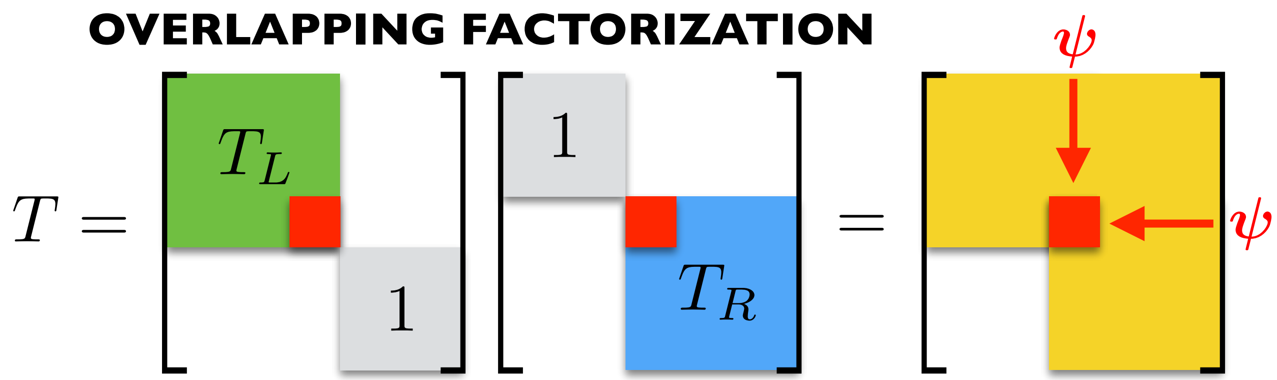

$$\Longrightarrow \quad f(z) = f_L(z) f_R(z)$$

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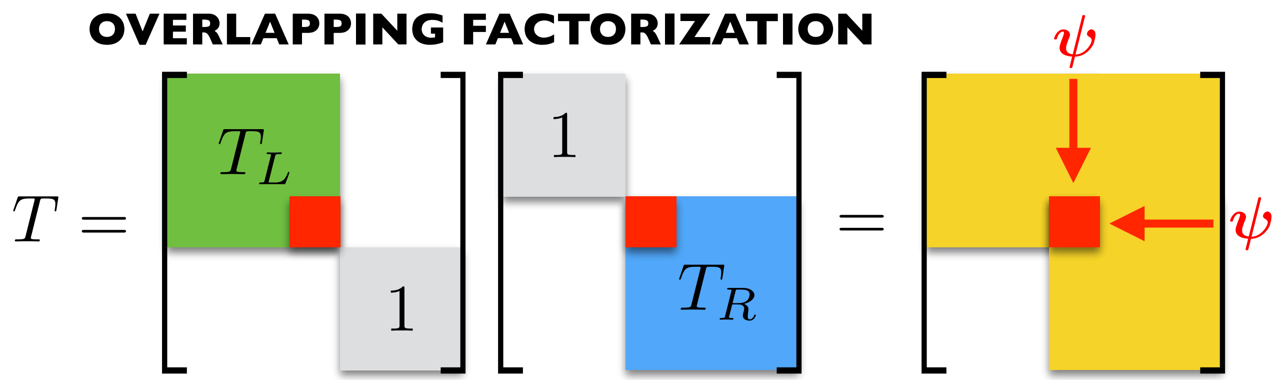

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When T is a **CMV matrix** this gives the known **OPUC Khrushchev's formula**

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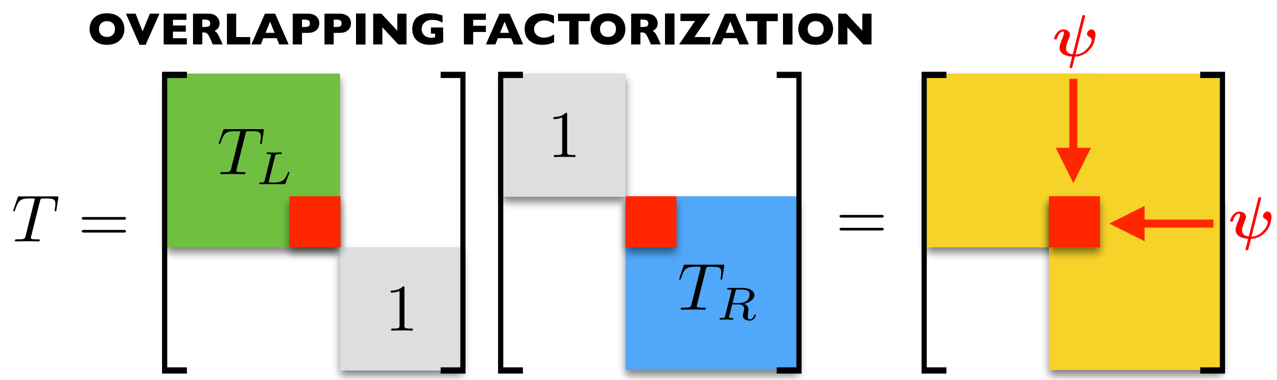

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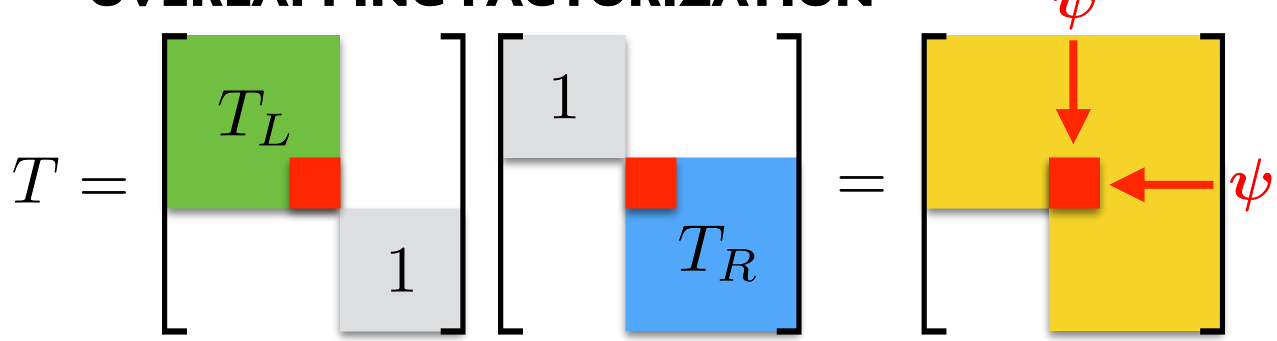

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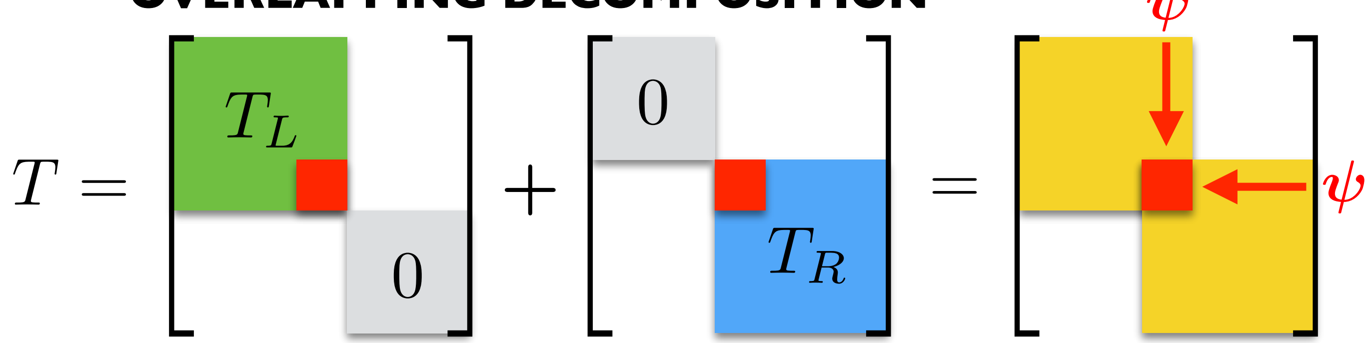

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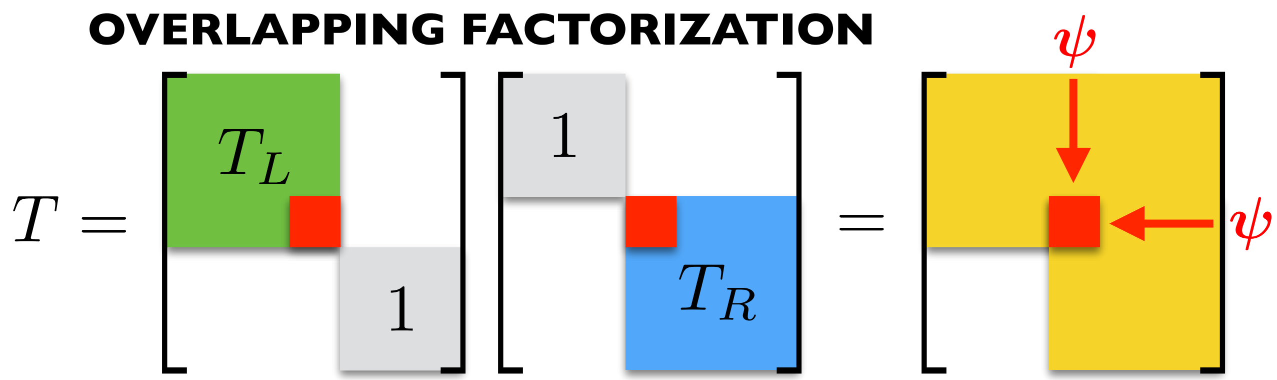

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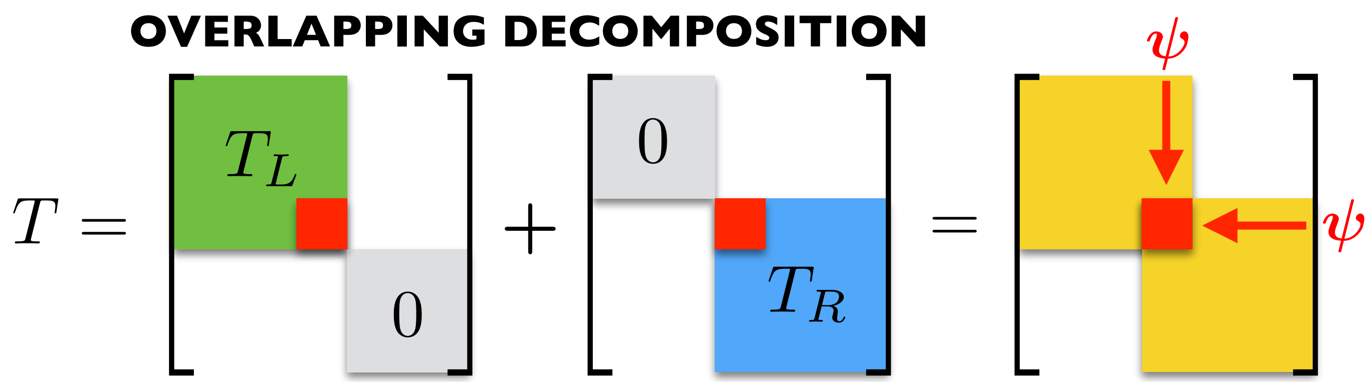

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DECOMPOSITION

When T is a **Jacobi matrix** this gives the **new OPRL Khrushchev's formula!**

General Khrushchev's Formulas

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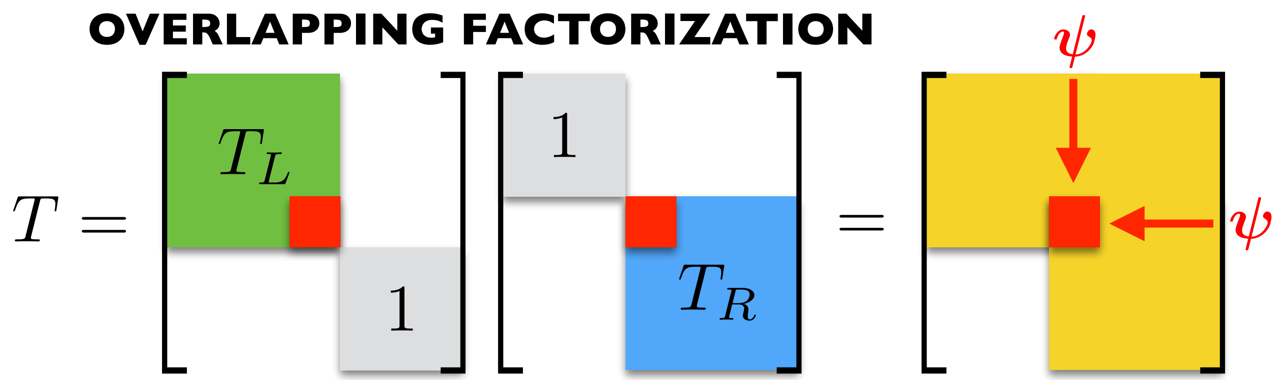

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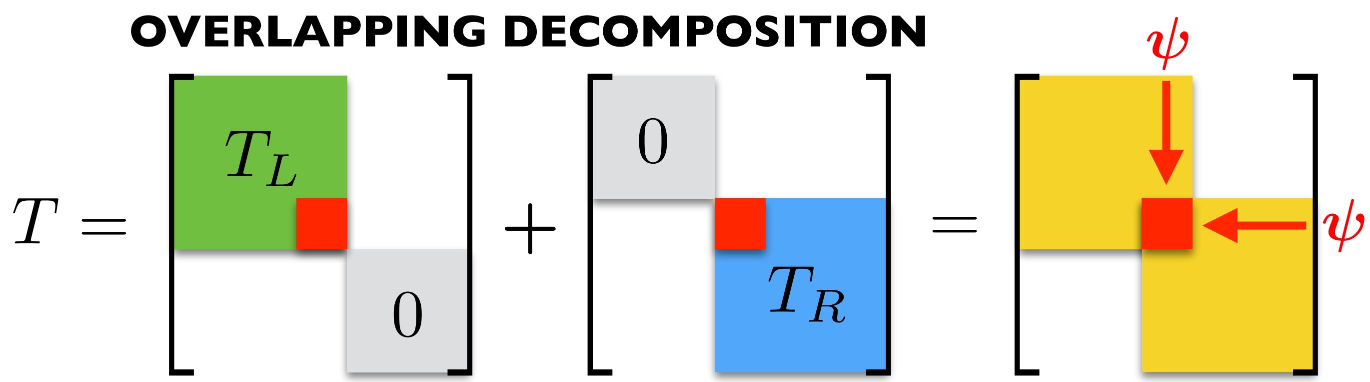

$$\Rightarrow \quad f(z) = f_L(z) + f_R(z)$$

DECOMPOSITION

When T is a **Jacobi matrix** this gives the **new OPRL Khrushchev's formula!**

Also valid for matrix valued case: **NEW Khrushchev's formulas for matrix OP!**