Analysis and approximation of fluids under singular forcing

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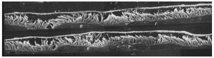
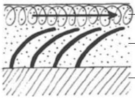
Outline

Outline

Motivation I: Active thin structures

- Motion of an incompressible viscous fluid.
- Active thin structures immersed in it.
- They exert a force supported on a lower dimensional object.





- The model becomes

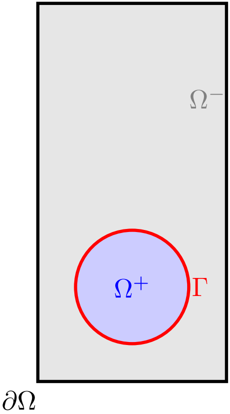
$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + \nabla \mathsf{p} = \mathbf{F}\delta_{\mathcal{Z}}, \quad \nabla\cdot\mathsf{u} = 0, \quad \text{in } \Omega, \qquad \mathsf{u} = \mathbf{0} \ \text{ on } \partial\Omega.$$

Here

- $\Omega \subset \mathbb{R}^d$ is the fluid domain with $d = 2$ or $d = 3$.
- u is the velocity, p is the presssure, $\mathbf{F}$ is a given forcing.
- $\boldsymbol{\varepsilon}(\mathsf{u}) = \frac{1}{2}\left(\nabla\mathsf{u} + \nabla\mathsf{u}^{\intercal}\right)$
- $\mathbb{S}$ is the stress tensor, e.g., $\mathbb{S} = 2\nu\boldsymbol{\varepsilon}$ gives the Stokes problem.
- $\mathcal{Z} \subset \bar{\Omega}$ with $0 \leq \dim\mathcal{Z} \leq d-1$.

Images shamelessly copied from:
Loïc Lacouture, *Ph.D Thesis*, Université Paris-Saclay, 2016.

Motivation II: Interface problems and immersed BMs, FEMs, etc.



- Fluid–solid interaction, two immiscible fluids separated by an interface, ...
- It reduces to an interface problem:

$$\nu = \nu_+ + (\nu_- - \nu_+)\,\chi_{\Omega^-}, \quad \mathsf{u} = \mathbf{0} \ \text{ in } \partial\Omega,$$

$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + \nabla\mathsf{p} = 0, \quad \nabla\cdot\mathsf{u} = 0, \quad \text{ in } \Omega^+ \cup \Omega^-,$$

and the interface conditions on Γ

$$[\![\mathsf{u}]\!] = \mathbf{0}, \qquad [\![(\mathbb{S} - \mathsf{p}\mathbb{I})\cdot\mathbf{n}]\!] = \boldsymbol{\sigma},$$

where $[\![\cdot]\!]$ is the jump, and $\boldsymbol{\sigma}$ can denote, for instance, surface tension.

- It reduces to

$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + \nabla\mathsf{p} = \boldsymbol{\sigma}\delta_\Gamma$$

where now $\mathbb{S}$ is may be discontinuous.

C. Peskin, Y. Mori, D. Boffi, L. Gastaldi, L. Heltai, T. Lin, P. Yue, ...

Motivation III: Generalized Smagorinsky models

- One of the first subgrid models of turbulence is due to Smagorinsky

$$\mathbb{S}(x, \boldsymbol{\varepsilon}) = 2\left(\nu + \nu_{NL}|\boldsymbol{\varepsilon}|\right)\boldsymbol{\varepsilon}.$$

- One of the main criticisms of this model is that it tends to overdissipate near walls.
- For this reason, several refinements and variations have been suggested. In particular.

$$\mathbb{S}(x, \boldsymbol{\varepsilon}) = 2\left(\nu + \nu_{NL}|\boldsymbol{\varepsilon}|\operatorname{dist}(x, \partial\Omega)^{\alpha}\right)\boldsymbol{\varepsilon}, \qquad \alpha \in (0, 2),$$

  where $\operatorname{dist}(\cdot, \partial\Omega)$ is the distance to the boundary. The idea is that the additional dissipation is dampened as one approaches the boundary.

- In summary, our model reads

$$-\nabla\cdot\mathbb{S}(x, \boldsymbol{\varepsilon}(\mathsf{u})) + (\mathsf{u}\cdot\nabla)\,\mathsf{u} + \nabla\mathsf{p} = \mathbf{f}, \quad \nabla\cdot\mathsf{u} = 0, \quad \text{in } \Omega, \qquad \mathsf{u} = \mathbf{0} \text{ on } \partial\Omega.$$

J. Smagorinsky, 1963.

Lesieur, 2008. Layton, 2016.

Sagaut, 2001. Vreman, 2003. Dunca et al., 2013. Berselli et al., 2006.

J. Rappaz and J. Rochat, CRAS 2016. J. Rappaz and J. Rochat, Comput. Methods Appl. Sci. 2019

Other non-Newtonian fluids under rough forcing

- We consider a nonlinear Stokes system

$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + \nabla \mathsf{p} = -\nabla\cdot\mathbf{F},$$

  with $\mathbf{F} \in \mathbf{L}^q(\Omega)$ with $q \in (1,\infty)$. We assume $\mathbb{S}$ satisfies:
  - It is Carathéodory.
  - For $\boldsymbol{\varepsilon} \in \mathbb{R}^{d\times d}$ and $x \in \Omega$ we have

$$|\boldsymbol{\varepsilon}|^2 - 1 \lesssim \mathbb{S}(x,\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon}, \qquad |\mathbb{S}(x,\boldsymbol{\varepsilon})| \le |\boldsymbol{\varepsilon}| + 1$$

  - It is linear at infinity: There is $\nu > 0$ such that, uniformly in x,

$$\lim_{|\boldsymbol{\varepsilon}|\to\infty} \frac{|\mathbb{S}(x,\boldsymbol{\varepsilon}) - 2\nu\boldsymbol{\varepsilon}|}{|\boldsymbol{\varepsilon}|} = 0$$

  - For $\boldsymbol{\varepsilon}_1 \neq \boldsymbol{\varepsilon}_2$ and uniformly in x

$$(\mathbb{S}(x,\boldsymbol{\varepsilon}_1) - \mathbb{S}(x,\boldsymbol{\varepsilon}_1)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) > 0, \qquad \lim_{|\boldsymbol{\varepsilon}|\to\infty} \left| \frac{\partial \mathbb{S}(x,\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - 2\nu\mathbb{I} \right| = 0$$

- Notice that, if $\mathbf{F} \notin \mathbf{L}^2(\Omega)$, u is not an admissible test function anymore. For this reason, we call this type of forcings rough.
- Such systems have been considered before[📘] under the assumption that $\Omega \in C^1$.

📘 M. Bulíček, J. Burczak, S. Schwarzacher, SIMAT 2016.

Outline

Outline

The guiding principle I: weighted spaces

- To motivate our approach consider, for $z \in \mathbb{R}^d$,

$$-\Delta \mathsf{u} + \nabla \mathsf{p} = \mathbf{F}\delta_z, \qquad \nabla\cdot \mathsf{u} = 0, \quad \text{in } \mathbb{R}^d,$$

  i.e., the fundamental solution to the Stokes problem.
- It is known that[📕]

$$|\nabla \mathsf{u}(x)| \approx |x-z|^{1-d}, \qquad |\mathsf{p}(x)| \approx |x-z|^{1-d}.$$

- Thus

$$|\nabla \mathsf{u}(x)|, |\mathsf{p}(x)| \notin L^2(E),$$

  but

$$\alpha \in (d-2,\infty) \implies \int_E |x-z|^\alpha \left(|\nabla \mathsf{u}(x)|^2 + |\mathsf{p}(x)|^2 \right) \mathrm{d}x < \infty,$$

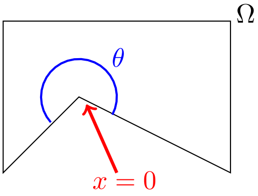
  for every compact E.

[📕] G.P. Galdi, 2011.

The guiding principle II: weighted spaces

- If Ω is bounded and smooth we expect a similar behavior of u and p.
- If Ω is only Lipschitz boundary singularities will appear. In fact, consider[1]

$$-\Delta u = f, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega$$



- It is known that even if $f \in C^\infty(\Omega)$; we have, with $s = \pi/\theta < 1$,

$$|\nabla u| \approx |x|^{s-1}, \qquad \alpha \in (-2, -2+2(1-s)) \implies \int_\Omega |x|^\alpha |\nabla u|^2 \, dx = \infty$$

- Our analysis then will consider two cases: Lipschitz domains and convex polyhedra with $d = 3$.

[1] Grisvard, 1985.

Muckenhoupt weights

Definition (Muckenhoupt weight)

Let $q \in [1,\infty)$. A function $0 \le \omega \in L^1_{loc}(\mathbb{R}^d)$ belongs to A_q if

$$[\omega]_{A_q} = \sup_B \frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x \left(\frac{1}{|B|}\int_B \omega^{1/(1-q)}(x)\,\mathrm{d}x\right)^{q-1} < \infty,$$
$$[\omega]_{A_1} = \sup_B \frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x \sup_{x\in B}\frac{1}{\omega(x)} < \infty.$$

Notice: that if $\omega \in A_q$, then $\omega' = \omega^{1/(1-q)} \in A_{q'}$.

- It is known that if $\mathcal{Z} \subset \mathbb{R}^d$ with $\dim \mathcal{Z} = k < d$, then $\mathrm{dist}(\cdot,\mathcal{Z})^\alpha \in A_q$ provided

$$-(d-k) < \alpha < (d-k)(q-1).$$

  Thus $|x-z|^\alpha \in A_2$ for $\alpha \in (-d,d)$.
- We will look for solutions in suitably weighted spaces!

The Stokes problem

- In summary $\Omega \subset \mathbb{R}^d$ is at least Lipschitz, we seek for (u, p) that solve

$$-\nu\Delta\mathsf{u} + \nabla\mathsf{p} = \mathbf{f}, \quad \nabla\cdot\mathsf{u} = 0, \ \text{ in } \Omega, \qquad \mathsf{u} = \mathbf{0}, \ \text{ on } \partial\Omega.$$

- The issue is that $\mathbf{f}$ is rough: e.g. $\mathbf{f} = \mathbf{F}\delta_z$ with $z \in \Omega$.

The functional setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain that is at least Lipschitz. Assume that, $q \in (1, \infty)$ $\varpi \in A_q$ and introduce the weighted spaces

$$L^q(\varpi, \Omega) = \left\{ v \in L^1_{loc}(\Omega) : \int_\Omega |v|^q \varpi \, \mathrm{d}x < \infty \right\},$$
$$W^{1,q}(\varpi, \Omega) = \left\{ v \in L^q(\varpi, \Omega) : \nabla v \in \mathbf{L}^q(\varpi, \Omega) \right\},$$
$$W^{1,q}_0(\varpi, \Omega) = \left\{ v \in W^{1,q}(\varpi, \Omega) : v = 0 \text{ on } \partial\Omega \right\}.$$

- Since $\varpi \in A_q$, these spaces satisfy most of the "usual properties".
- As usual $\mathbf{L}^q(\varpi, \Omega) = L^q(\varpi, \Omega)^d$.
- We will look for:

$$\boxed{\mathsf{u} \in \mathbf{W}^{1,q}_0(\varpi, \Omega) \text{ and } \mathsf{p} \in L^q(\varpi, \Omega)/\mathbb{R},}$$

for a suitable $q \in (1, \infty)$ and $\varpi \in A_q$.

Outline

The class $A_q(\Omega)$

- Assume the singular source is supported on $\mathcal{Z} \Subset \Omega$.

Definition (class $A_q(\Omega)$)

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. We say that $\omega \in A_q$ belongs to $A_q(\Omega)$ if there is an open set $\mathcal{G} \subset \Omega$, and positive constants $\delta > 0$ and $\omega_l > 0$ such that:

1. $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\} \subset \mathcal{G}$,
2. $\omega|_{\bar{\mathcal{G}}} \in C(\bar{\mathcal{G}})$, and
3. $\omega_l \leq \omega(x)$ for all $x \in \bar{\mathcal{G}}$.

Notice that:

- $|x - z|^\alpha \in A_2(\Omega)$ for $\alpha \in (-d, d)$.
- More generally, if $\mathcal{Z} \Subset \Omega$ with $\dim \mathcal{Z} = k < d$ then $\operatorname{dist}(x, \mathcal{Z})^\alpha \in A_2(\Omega)$ for $\alpha \in (-(d-k), (d-k))$.

Generalized saddle point formulation

Define the bilinear forms

$$a : \mathbf{H}_0^1(\varpi,\Omega) \times \mathbf{H}_0^1(\varpi^{-1},\Omega) \to \mathbb{R}$$
$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} \, \mathrm{d}x$$

and

$$b_\pm : \mathbf{H}_0^1(\varpi^{\pm 1},\Omega) \times L^2(\varpi^{\mp 1},\Omega) \to \mathbb{R}$$
$$b_\pm(\mathbf{v},q) = -\int_\Omega q \nabla\cdot \mathbf{v} \, \mathrm{d}x$$

Problem: Given $\mathbf{f} \in \mathbf{H}_0^1(\varpi^{-1},\Omega)'$ find $(\mathsf{u},\mathsf{p}) \in \mathbf{H}_0^1(\varpi,\Omega) \times L^2(\varpi,\Omega)/\mathbb{R}$ such that

$$\begin{cases} a(\mathsf{u},\mathbf{v}) + b_-(\mathbf{v},\mathsf{p}) = \mathbf{f}(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{H}_0^1(\varpi^{-1},\Omega), \\ b_+(\mathsf{u},q) = 0, & \forall q \in L^2(\varpi^{-1},\Omega)/\mathbb{R}. \end{cases}$$

For instance $z \in \Omega$ and $\mathbf{f} = \mathbf{F}\delta_z$. Then:

- $\varpi(x) = |x-z|^\alpha \in A_2(\Omega)$ for $\alpha \in (-d,d)$.
- $\delta_z \in H_0^1(\varpi^{-1},\Omega)'$ for $\alpha \in (d-2,d)$.

Well-posedness

Theorem

Let Ω be a Lipschitz domain and $\varpi \in A_2(\Omega)$. For every $\mathbf{f} \in \mathbf{H}_0^1(\varpi^{-1}, \Omega)'$ there are unique $(\mathsf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\varpi, \Omega) \times L^2(\varpi, \Omega)/\mathbb{R}$ that solve the generalized saddle point formulation. This solution satisfies

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\varpi,\Omega)} + \|\mathsf{p}\|_{L^2(\varpi,\Omega)/\mathbb{R}} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\varpi^{-1},\Omega)'}.$$

where the hidden constant is independent of u, p and $\mathbf{f}$.

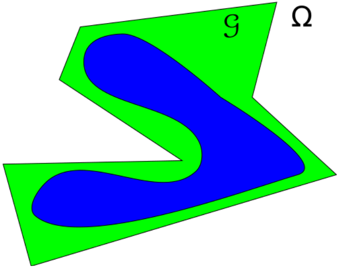
Remark

There is $\epsilon \in (0,1)$ that depends only on Ω such that if $|q-2| < \epsilon$, the problem is still well-posed in $\mathbf{W}_0^{1,q}(\varpi, \Omega) \times L^q(\varpi, \Omega)/\mathbb{R}$ for $\varpi \in A_q(\Omega)$ and $\mathbf{f} \in \mathbf{W}_0^{1,q'}(\varpi', \Omega)'$.

E. Otárola, AJS, JMAA 2019

Idea of the proof of well-posedness

- The result is true for C^1 domains and all $\varpi \in A_2$. [1]
- In addition the result is true for $C^{0,1}$ domains and $\varpi \equiv 1$. [2]
- WLOG we can assume that $\partial(\Omega \setminus \mathcal{G})$ is C^1.
- Glue the two previous results: Use Bulíček in $\Omega \setminus \mathcal{G}$ and Mitrea in $\mathcal{G}$.



Details

[1] M. Bulíček, J. Burczak, S. Schwarzacher, SIMAT 2016.

[2] M. Mitrea, M. Wright, Astérisque 2012.

The stationary Navier Stokes problem

- The stationary Navier Stokes problem: find (u, p) that solve

$$-\nu\Delta \mathsf{u} + (\mathsf{u}\cdot\nabla)\mathsf{u} + \nabla \mathsf{p} = \mathbf{f}, \quad \nabla\cdot\mathsf{u} = 0, \quad \text{in } \Omega, \qquad \mathsf{u} = \mathbf{0}, \quad \text{on } \partial\Omega.$$

Corollary ()

Let $d = 2$, Ω be Lipschitz, $\varpi \in A_2(\Omega)$, and $\mathbf{f} \in \mathbf{H}_0^1(\varpi^{-1}, \Omega)'$. The Navier Stokes problem has a solution $(\mathsf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\varpi, \Omega) \times L^2(\varpi, \Omega)/\mathbb{R}$. This solution satisfies

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\varpi,\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\varpi^{-1},\Omega)'}.$$

If, in addition, either $\mathbf{f}$ is sufficiently small, or $\nu > 0$ sufficiently big, then the solution is unique.

Proof.

In two dimensions, for $\varpi \in A_2(\Omega)$, we have $H^1(\varpi, \Omega) \hookrightarrow\hookrightarrow L^4(\varpi, \Omega)$ so that

$$\left|\int_\Omega \mathsf{u} \otimes \mathsf{u} : \nabla \mathbf{v} \, dx\right| = \left|\int_\Omega \varpi^{1/4}\mathsf{u} \otimes \varpi^{1/4}\mathsf{u} : \varpi^{-1/2}\nabla \mathbf{v} \, dx\right| \leq \|\mathsf{u}\|^2_{\mathbf{L}^4(\varpi,\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\varpi^{-1},\Omega)}.$$

The rest of the proof is by the usual fixed point arguments. □

E. Otárola, AJS, AML 2020.

Outline

Is Lipschitz good enough?

The previous results are nice, but:

- There is a restricted range of integrability: The Stokes problem is well posed for $q \in (2-\epsilon, 2+\epsilon)$, $\varpi \in A_q(\Omega)$ and $\mathbf{f} \in \mathbf{W}_0^{1,q}(\varpi', \Omega)'$. What if our problem requires a q outside of that range?
- What if the singular source touches the boundary?
- Recall the generalization of Smagorinsky:

$$\mathbb{S}(x, \boldsymbol{\varepsilon}) = 2\left(\nu + \nu_{NL}|\boldsymbol{\varepsilon}| \operatorname{dist}(x, \partial\Omega)^{\alpha}\right)\boldsymbol{\varepsilon}, \qquad \alpha \in [0, 2).$$

  The natural framework here is

$$\mathsf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{W}^{1,3}(\operatorname{dist}(\cdot, \partial\Omega)^{\alpha}, \Omega),$$
$$\mathsf{p} \in L^2(\Omega)/\mathbb{R} + L^{3/2}(\operatorname{dist}(\cdot, \partial\Omega)^{-\alpha/2}, \Omega)/\mathbb{R}.$$

  However

$$\operatorname{dist}(\cdot, \partial\Omega)^{\alpha} \in A_3 \setminus A_3(\Omega).$$

- ...

The Green matrix

The solution to the Stokes problem

$$-\nu\Delta \mathsf{u} + \nabla \mathsf{p} = -\nabla\cdot\mathbf{f}, \qquad \nabla\cdot\mathsf{u} = g, \quad \text{in } \Omega, \qquad \mathsf{u} = \mathbf{0}, \ \text{on } \partial\Omega$$

has the representation

$$\mathsf{u}_j(\xi) = \frac{1}{\nu}\langle \mathbf{f}, \nabla\mathbf{G}_j(\cdot,\xi)\rangle - \int_\Omega \lambda_j(x,\xi) g(x)\,\mathrm{d}x$$

where

$$\mathbb{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 & \mathbf{G}_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}$$

is the Green matrix.
The pairs $(\mathbf{G}_j, \lambda_j)$ solve

$$\begin{cases} -\Delta_x \mathbf{G}_j(x,\xi) + \nabla_x \lambda_j(x,\xi) = \delta(x-\xi)\mathbf{e}_j, \\ \nabla\cdot_x \mathbf{G}_j(x,\xi) = 0, \\ \mathbf{G}_j(x,\xi) = \mathbf{0} \ x \in \partial\Omega \end{cases} \quad j = 1,\ldots,3 \qquad \begin{cases} -\Delta_x \mathbf{G}_4(x,\xi) + \nabla_x \lambda_4(x,\xi) = \mathbf{0}, \\ \nabla\cdot_x \mathbf{G}_j(x,\xi) = \delta(x-\xi) - \phi(x), \\ \mathbf{G}_j(x,\xi) = \mathbf{0} \ x \in \partial\Omega \end{cases}$$

where $\phi \in C_0^\infty(\Omega)$ is such that $\int_\Omega \phi(x)\,\mathrm{d}x = 1$ and we normalize

$$\int_\Omega \lambda_j(x,\xi)\phi(x)\,\mathrm{d}x = 0, \quad j = 1,\ldots,4.$$

The Green matrix: Mixed derivative estimates on convex polyhedra

Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedron. Then[📖] there is $\sigma \in (0,1)$ such that for all $\alpha, \beta \in \mathbb{N}_0^3$

$$\left|\partial_x^\alpha \partial_\xi^\beta \mathbb{G}_{i,j}(x,\xi) - \partial_y^\alpha \partial_\xi^\beta \mathbb{G}_{i,j}(y,\xi)\right| \lesssim |x-y|^\sigma (|x-\xi|^{-a} + |y-\xi|^{-a})$$

$$\left|\partial_x^\alpha \partial_\xi^\beta \mathbb{G}_{i,j}(x,\xi) - \partial_y^\alpha \partial_\xi^\beta \mathbb{G}_{i,j}(x,\eta)\right| \lesssim |\xi-\eta|^\sigma (|x-\xi|^{-a} + |x-\eta|^{-a})$$

whenever $|\alpha| \le 1 - \delta_{i,4}$, $|\beta| \le 1 - \delta_{j,4}$ and

$$a = 1 + \sigma + \delta_{i,4} + \delta_{j,4} + |\alpha| + |\beta|.$$

- In particular, for $j = 1, \ldots, 3$,

$$|\partial_{x_k}\partial_{\xi_\ell}\mathbf{G}_j(x,\xi) - \partial_{x_k}\partial_{\xi_\ell}\mathbf{G}_j(x,\eta)| \lesssim |\xi-\eta|^\sigma (|x-\xi|^{-3-\sigma} + |x-\eta|^{-3-\sigma}),$$
$$|\partial_{\xi_\ell}\lambda_j(x,\xi) - \partial_{\xi_\ell}\lambda_j(x,\eta)| \lesssim |\xi-\eta|^\sigma (|x-\xi|^{-3-\sigma} + |x-\eta|^{-3-\sigma}).$$

[📖] J. Rossman, Rostock Math. Kolloq 2010.

Well-posedness

Theorem (📘)

Let $\Omega \subset \mathbb{R}^3$ *be a convex polyhedron,* $q \in (1,\infty)$, $\varpi \in A_q$, $\mathbf{f} \in \mathbf{L}^q(\varpi,\Omega)$, *and* $g \in L^q(\varpi,\Omega)/\mathbb{R}$. *Then, there are unique* $(\mathsf{u},\mathsf{p}) \in \mathbf{W}_0^{1,q}(\varpi,\Omega) \times L^q(\varpi,\Omega)/\mathbb{R}$ *that solve the generalized saddle point formulation. This solution satisfies*

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|\mathsf{p}\|_{L^q(\varpi,\Omega)/\mathbb{R}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

where the hidden constant is independent of $\mathsf{u},\mathsf{p},\mathbf{f}$ *and* g.

📘 E. Otárola, AJS, arXiv 2021.

Idea of the proof of well-posedness

- The pointwise estimates of the mixed derivatives allow us to treat the solution representation as a singular integral operator of CZ type.
- Oscillation estimate: for $s > 1$

$$\mathcal{M}^{\sharp}_{\Omega}\left[\nabla \mathsf{u}\right](z) \lesssim \mathcal{M}\left[|\mathbf{f}|^s\right](z)^{1/s} + \mathcal{M}\left[|g|^s\right](z)^{1/s}.$$

- Weighted Fefferman-Stein inequality[1]

$$\left\| \nabla \mathsf{u} - \frac{1}{|\Omega|}\int_{\Omega} \nabla \mathsf{u}\, \mathrm{d}x \right\|_{\mathbf{L}^q(\varpi,\Omega)} \leq \|\mathcal{M}^{\sharp}_{\Omega}\left[\nabla \mathsf{u}\right]\|_{\mathbf{L}^q(\varpi,\Omega)}$$

- Continuity of maximal function on weighted spaces

$$\left\|\mathcal{M}\left[|\mathbf{f}|^s\right]^{1/s}\right\|_{L^q(\varpi,\Omega)} + \left\|\mathcal{M}\left[|g|^s\right]^{1/s}\right\|_{L^q(\varpi,\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

- Pressure estimate: Using the surjectivity of the Bogovskiĭ operator[2]

$$\|\mathsf{p}\|_{L^q(\varpi,\Omega)} \lesssim \sup_{\mathbf{v}\in \mathbf{W}^{1,q'}_0(\varpi',\Omega)} \frac{\int_{\Omega} \mathsf{p}\nabla\cdot\mathbf{v}\,\mathrm{d}x}{\|\nabla \mathbf{v}\|_{\mathbf{L}^{q'}(\varpi',\Omega)}}.$$

Details

[1] L. Diening, M. Růžička, K. Schumacher, 2010.
[2] Acosta and Durán, 2017.

Generalized Smagorisnsky models I

- Recall that the generalized Smagorinsky model read

$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + (\mathsf{u}\cdot\nabla)\mathsf{u} + \nabla\mathsf{p} = -\nabla\cdot\mathbf{f},$$

where

$$\mathbb{S}(x,\boldsymbol{\varepsilon}) = 2\left(\nu + \nu_{NL}|\boldsymbol{\varepsilon}|\operatorname{dist}(x,\partial\Omega)^{\alpha}\right)\boldsymbol{\varepsilon}, \quad \alpha\in[0,2).$$

- For $\alpha\in(-1,2)$

$$\operatorname{dist}(x,\partial\Omega)^{\alpha}\in A_3.$$

- We seek for solutions

$$\mathsf{u}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{W}_0^{1,3}(\operatorname{dist}(\cdot,\partial\Omega)^{\alpha},\Omega)$$
$$\mathsf{p}\in L^2(\Omega)/\mathbb{R} + L^{3/2}(\operatorname{dist}(\cdot,\partial\Omega)^{-\alpha/2},\Omega)/\mathbb{R}.$$

Generalized Smagorisnsky models II

Theorem (📘)

Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedron and $\alpha \in (-1, 2)$. If

$$\mathbf{f} \in \mathbf{L}^2(\Omega) + \mathbf{L}^{3/2}(\mathrm{dist}(\cdot, \partial\Omega)^{-\alpha/2}, \Omega)$$

Then the generalized Smagorinksy model has a solution (u, p). If, in addition ν is sufficiently large, or $\mathbf{f}$ sufficiently small, then u is unique.

Proof.

- Minimize the energy

$$\mathcal{J}(\mathbf{v}) = \frac{\nu}{2} \int_\Omega |\boldsymbol{\varepsilon}(\mathbf{v})|^2 \,\mathrm{d}x + \frac{2\nu_{NL}}{3} \int_\Omega \mathrm{dist}(x, \partial\Omega)^\alpha |\boldsymbol{\varepsilon}(\mathbf{v})|^3 \,\mathrm{d}x - \int_\Omega \mathbf{f} : \nabla \mathbf{v} \,\mathrm{d}x.$$

- Usual tricks for convection.
- Two pressures: unweighted inf-sup ($L^2(\Omega)$) and weighted one ($L^{3/2}(\mathrm{dist}(\cdot, \partial\Omega)^{-\alpha/2}, \Omega)$). □

📘 Otárola, AJS, arXiv 2021.

Generalized Smagorisnsky models III

- Notice that $\mathrm{dist}(\cdot, \partial\Omega)^\alpha \notin A_q(\Omega)$, for any q, whenever $\alpha \neq 0$.
- Even without convection p is unique only if

$$\alpha \leq \frac{1}{2} \implies L^2(\Omega) \hookrightarrow L^{3/2}(\mathrm{dist}(\cdot, \partial\Omega)^{-\alpha/2}, \Omega).$$

- Slight generalization: Let $q \in (1, \infty)$, $\omega \in A_q$, and $\mathbf{f} \in \mathbf{L}^2(\Omega) + \mathbf{L}^{q'}(\varpi', \Omega)$, then

$$\mathsf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{W}_0^{1,q}(\varpi, \Omega),$$
$$\mathsf{p} \in L^2(\Omega)/\mathbb{R} + L^{q'}(\varpi', \Omega)/\mathbb{R}$$

Other non-Newtonian fluids I

- Consider now

$$-\nabla\cdot\mathbb{S}(x,\boldsymbol{\varepsilon}(\mathsf{u})) + \nabla\mathsf{p} = -\nabla\cdot\mathbf{f},\ \nabla\cdot\mathsf{u} = g \ \text{ in } \Omega, \qquad \mathsf{u} = \mathbf{0}, \ \text{ on } \partial\Omega,$$

with $\mathbb{S}$ "linear at infinity".

Theorem ([1])

Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedron, $q \in (1,\infty)$, and $\varpi \in A_q$. If

$$\mathbf{f} \in \mathbf{L}^q(\varpi,\Omega), \qquad g \in L^q(\varpi,\Omega)/\mathbb{R}$$

Then the problem has a unique solution

$$(\mathsf{u},\mathsf{p}) \in \mathbf{W}_0^{1,q}(\varpi,\Omega) \times L^q(\varpi,\Omega)/\mathbb{R},$$

which satisfies the estimate

$$\|\nabla\mathsf{u}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|\mathsf{p}\|_{L^q(\varpi,\Omega)/\mathbb{R}} \lesssim 1 + \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

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[1] Otárola, AJS, arXiv 2021.

Idea of the proof

- Follow the proof for C^1 domains [📖].
- Properties of weights: If, for some $s \in (1,2]$,

$$(\mathsf{u},\mathsf{p}) \in \mathbf{W}^{1,s}_0(\Omega) \times L^s(\Omega)/\mathbb{R}$$

then $(\mathsf{u},\mathsf{p}) \in \mathbf{H}^1_0(\tilde{\varpi}_j,\Omega) \times L^2(\tilde{\varpi}_j,\Omega)/\mathbb{R}$ with

$$\tilde{\varpi}_j = \min\left\{\varpi, j\mathcal{M}\left[\nabla \mathsf{u}\right]^{s-2}, j\mathcal{M}\left[\mathsf{p}\right]^{s-2}\right\}.$$

- Key step: Represent (u,p) as the solution to a Stokes problem so that

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\tilde{\varpi}_j,\Omega)} + \|\mathsf{p}\|_{L^2(\tilde{\varpi}_j,\Omega)} \lesssim 1 + \|\mathbf{f}\|_{\mathbf{L}^2(\tilde{\varpi}_j,\Omega)} + \|g\|_{L^2(\varpi_j,\Omega)},$$

uniformly in j. Pass to the limit $j \to \infty$. Two important points here are:

  - Asymptotic linearity: This allows to "absorb" nonlinear terms.
  - Convexity: There is no information on the behavior of u or p near the boundary. Thus,

$$\tilde{\varpi}_j \notin A_2(\Omega).$$

[📖] M. Bulíček, J. Burczak, S. Schwarzacher, SIMAT 2016.

Outline

Outline

Discretization

- $\mathscr{T} = \{T\}$ is a conforming and shape regular partition of $\bar{\Omega}$ into simplices of size $h_T = \operatorname{diam}(T)$.
- Set $h_{\mathscr{T}} = \max h_T$.
- $\mathcal{V}(\mathscr{T})$ is the FE velocity space, $\mathcal{P}(\mathscr{T})$ is the pressure space and we assume that they are inf-sup stable in the classical sense.
- Since, for any $q \in (1, \infty)$ and $\varpi \in A_q$

$$\begin{aligned}\mathcal{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T}) &\subset \mathbf{W}_0^{1,\infty}(\Omega) \times L^\infty(\Omega)/\mathbb{R} \\ &\subset \mathbf{W}_0^{1,q}(\varpi, \Omega) \times L^q(\varpi, \Omega)/\mathbb{R},\end{aligned}$$

given

$$(\mathsf{u}, \mathsf{p}) \in \mathbf{W}_0^{1,q}(\varpi, \Omega) \times L^q(\varpi, \Omega)/\mathbb{R}$$

we define its Stokes projection to be the pair

$$(\mathbf{u}_{\mathscr{T}}, p_{\mathscr{T}}) \in \mathcal{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$$

such that

$$\begin{cases} a(\mathsf{u} - \mathbf{u}_{\mathscr{T}}, \mathbf{v}_{\mathscr{T}}) + b_-(\mathbf{v}_{\mathscr{T}}, \mathsf{p} - p_{\mathscr{T}}) = 0, & \forall \mathbf{v}_{\mathscr{T}} \in \mathcal{V}(\mathscr{T}), \\ b_+(\mathsf{u} - \mathbf{u}_{\mathscr{T}}, q_{\mathscr{T}}) = 0, & \forall q_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}). \end{cases}$$

Stability

Lemma (discrete inf-sup)

Let $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$ be Lipschitz, $\mathscr{T}$ be quasiuniform, $q \in (1, \infty)$ and $\varpi \in A_q$. Then,

$$\|r_{\mathscr{T}}\|_{L^q(\varpi,\Omega)} \lesssim \sup_{\mathbf{v}_{\mathscr{T}} \in \mathcal{V}(\mathscr{T})} \frac{\int_\Omega \nabla \cdot \mathbf{v}_{\mathscr{T}} r_{\mathscr{T}} \, \mathrm{d}x}{\|\nabla \mathbf{v}_{\mathscr{T}}\|_{\mathbf{L}^{q'}(\varpi',\Omega)}}, \quad \forall r_{\mathscr{T}} \in \mathcal{P}(\mathscr{T}),$$

where the hidden constant does not depend on $h_{\mathscr{T}}$.

Theorem (stability)

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a convex polytope. Let $q \in (1, \infty)$ and

- $q \geq 2$ $\varpi \in A_{q/2}$,
- $q \in (1, 2]$ $\varpi' \in A_{q'/2}$.

If $\mathscr{T}$ is quasiuniform, then

$$\|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|p_{\mathscr{T}}\|_{L^q(\varpi,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|\mathsf{p}\|_{L^q(\varpi,\Omega)},$$

where the constant is independent of $h_{\mathscr{T}}$, u, and p.

R.G. Durán, E. Otárola, AJS, Math. Comp. 2020.

Idea of the proof of stability I

- The pressure estimate follows form the discrete inf-sup condition.
- The case $q < 2$ follows by duality.
- The case $q > 2$ follows from *Rubio de Francia* extrapolation: If

$$T : L^2(\rho, \Omega) \to L^2(\rho, \Omega)$$

  boundedly for all $\rho \in A_1$, then

$$T : L^q(\varpi, \Omega) \to L^q(\varpi, \Omega)$$

  boundedly for all $\varpi \in A_{q/2}$.
- It remains then to show, for $\varpi \in A_1$,

$$\|\nabla \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\varpi,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^2(\varpi,\Omega)} + \|\mathsf{p}\|_{L^2(\varpi,\Omega)},$$

Idea of the proof of stability II

- We use the approximate Green's matrix $\tilde{\mathbf{G}}$ and its approximation $\mathbf{G}_{\mathscr{T}}$ to represent, for $z \in T \in \mathscr{T}$

$$\partial_i \mathbf{u}^j_{\mathscr{T}}(z) = a(\mathsf{u}, \mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}) + b_-(\mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}, \mathsf{p}) + \int_\Omega \tilde{\delta}_z \partial_i \mathsf{u}^j \,\mathrm{d}x$$

- Thus, with $\mathbf{E} = \mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}$

$$\int_\Omega \varpi |\partial_i \mathbf{u}^j_{\mathscr{T}}|^2 \,\mathrm{d}x \lesssim \int_\Omega \varpi \left[\int_\Omega \nabla \mathsf{u} : \nabla \mathbf{E} \,\mathrm{d}x \right]^2 \mathrm{d}z + \int_\Omega \varpi \left[\int_\Omega \mathsf{p} \nabla \cdot \mathbf{E} \,\mathrm{d}x \right]^2 \mathrm{d}x + \int_\Omega \varpi \left[\frac{1}{|T|} \int_T \partial_i \mathsf{u}^j \,\mathrm{d}x \right]^2 \mathrm{d}z.$$

- Properties of $\mathbf{E}$ 📘 and the fact that $\varpi \in A_1$ then yield the result.

Details

📘 V. Girault, R.H. Nochetto, R. Scott, Num. Math. 2015.

Outline

An error estimate in L^q

Corollary

In the setting of the previous result, if $q > 2$

$$\|\mathsf{u} - \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^q(\Omega)} \lesssim h_{\mathscr{T}}^{1+d/q} \varpi(\mathscr{T})^{-1/q} \left(\|\nabla \mathsf{u}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|\mathsf{p}\|_{L^q(\varpi,\Omega)} \right),$$

where

$$\varpi(\mathscr{T}) = \sup_{T \in \mathscr{T}} \varpi(T), \qquad \varpi(T) = \int_T \varpi \, \mathrm{d}x.$$

In particular, if the forcing is $\mathbf{F}\delta_z$ we have, for any $\epsilon > 0$,

$$\|\mathsf{u} - \mathbf{u}_{\mathscr{T}}\|_{\mathbf{L}^2(\Omega)} \lesssim h_{\mathscr{T}}^{2-d/2-\epsilon}.$$

Proof.

A duality argument. □

R.G. Durán, E. Otárola, AJS, Math. Comp. 2020.

Generalized Smagorinsky models

- Consider the generalized Smagorinsky model. $\Omega \subset \mathbb{R}^3$ is a convex polyhedron.
- No convection.
- (u, p) is the exact solution, $(\mathsf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$ is its Galerkin approximation, and $(\mathbf{u}_{\mathscr{T}}, p_{\mathscr{T}})$ is its Stokes projection.

Corollary (📘)

Assume that $\mathscr{T}$ is quasiuniform, and that $\alpha \in (-1, 1/2)$. Then, the pair $(\mathsf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$ exists, is unique, and stable. Moreover,

$$\|\boldsymbol{\varepsilon}(\mathsf{u} - \mathsf{u}_{\mathscr{T}})\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\varepsilon}(\mathsf{u} - \mathsf{u}_{\mathscr{T}})\|^3_{\mathbf{L}^3(\mathrm{dist}(\cdot,\partial\Omega)^\alpha,\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\mathsf{u} - \mathbf{u}_{\mathscr{T}})\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\varepsilon}(\mathsf{u} - \mathbf{u}_{\mathscr{T}})\|^{3/2}_{\mathbf{L}^3(\mathrm{dist}(\cdot,\partial\Omega)^\alpha,\Omega)}.$$

Proof.

Repeat the old arguments for the p–Laplacian[📕].
The restriction $\alpha \in (-1, 1/2)$ guarantees that $\mathrm{dist}(\cdot, \partial\Omega)^\alpha \in A_{3/2}$. □

📘 E. Otárola, AJS. arXiv 2021

📕 Glowinski, Marrocco. RAIRO 1975. Ciarlet book 1978. S.-S. Chow, Numer. Math. 1989.

Other non-Newtonian fluids

- Consider the "linear at infinity" models.
- $\Omega \subset \mathbb{R}^3$ is a convex polyhedron, $q \in (1, \infty)$, $\varpi \in A_{q/2}$, $\mathbf{f} \in \mathbf{L}^q(\varpi, \Omega)$, and $g = 0$.
- (u, p) is the exact solution, $(\mathsf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$ is its Galerkin approximation, and $(\mathbf{u}_{\mathscr{T}}, p_{\mathscr{T}})$ is its Stokes projection.

Theorem (📘)

If $\mathscr{T}$ is quasiuniform the pair $(\mathsf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$ exists, is unique, and stable. Moreover, up to subsequences, in $\mathbf{W}_0^{1,q}(\varpi, \Omega)$

$$\mathsf{u}_{\mathscr{T}} \rightharpoonup \mathsf{u}, \qquad h_{\mathscr{T}} \to 0.$$

Proof.

- Finite dimensions $\implies$ Existence and uniquenes.
- Stability of the Stokes projection $\implies$ stability of $(\mathsf{u}_{\mathscr{T}}, \mathsf{p}_{\mathscr{T}})$.
- Convergence by compactness. We require Minty's trick, and a Fortin operator in weighted spaces (discrete inf-sup). □

📘 E. Otárola, AJS. arXiv 2021

Outline

A posteriori error estimation

- Since we are trying to approximate rough objects we need to consider a posteriori error estimators.
- Consider the Stokes problem with forcing $\mathbf{F}\delta_z$ and $z \in \Omega$. Define

$$D_T = \max_{x \in T} |x - z|, \qquad T \in \mathscr{T}.$$

- The local error indicator, for $T \in \mathscr{T}$, is

$$\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, p_\mathscr{T}; T)^2 = h_T^2 D_T^\alpha \|\Delta \mathbf{u}_\mathscr{T} - \nabla p_\mathscr{T}\|^2_{\mathbf{L}^2(T)} + \|\nabla \cdot \mathbf{u}_\mathscr{T}\|^2_{L^2(\mathrm{dist}_z^\alpha, T)} + h_T D_T^\alpha \|[\![(\nabla \mathbf{u}_\mathscr{T} - p_\mathscr{T} \mathbb{I}) \cdot \mathbf{n}]\!]\|^2_{\mathbf{L}^2(\partial T \setminus \partial\Omega)} + h_T^{\alpha+2-d} |\mathbf{F}|^2 \#(T \cap \{z\}),$$

where, as usual, $\mathbf{n}$ is the normal to ∂T and $[\![w]\!]$ denotes the jump of w.
- The error estimator, as expected, is then

$$\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, p_\mathscr{T}, \mathscr{T}) = \|\{\mathcal{E}_\alpha(\mathbf{u}_\mathscr{T}, p_\mathscr{T}; T)\}\|_{\ell^2(\mathscr{T})}.$$

A posteriori error estimation: Reliability

From now on $(\mathbf{e}_u, e_p) = (\mathsf{u} - \mathbf{u}_{\mathscr{T}}, \mathsf{p} - p_{\mathscr{T}})$. We have

Theorem ([1])

If $\alpha \in (d-2, d)$ *then*

$$\|\nabla \mathbf{e}_u\|_{\mathbf{L}^2(\mathrm{dist}_z^\alpha, \Omega)} + \|\mathbf{e}_p\|_{L^2(\mathrm{dist}_z^\alpha, \Omega)} \lesssim \mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, p_{\mathscr{T}}, \mathscr{T}),$$

where the hidden constant is independent of the continuous and discrete solutions, $h_{\mathscr{T}}$ *and* $\#\mathscr{T}$.

Proof.

- Usual "disintegration by parts argument" + Galerkin orthogonality.
- The existence of an interpolation operator $\Pi_{\mathscr{T}} : \mathbf{L}^1(\Omega) \to \mathcal{V}(\mathscr{T})$ that plays nice with weighted norms[2].
- A bound on $\|\delta_z\|'_{H^1(\mathrm{dist}_z^{-\alpha}, T)}$ for $z \in T$[3].

□

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[1] A. Allendes, E. Otárola, AJS. CMAME 2019.

[2] R.H. Nochetto, E. Otárola and AJS. Numer. Math. 2016.

[3] J.P. Agnelli, E. Garau, P. Morin. M2AN 2014.

A posteriori error estimation: Local efficiency I

Theorem (📘)

If $\alpha \in (d-2, d)$ and $T \in \mathscr{T}$ then

$$\mathcal{E}_\alpha(\mathbf{u}_{\mathscr{T}}, p_{\mathscr{T}}; T)^2 \lesssim \|\nabla \mathbf{e}_u\|^2_{\mathbf{L}^2(\mathrm{dist}_z^\alpha, \mathcal{N}_T)} + \|e_p\|^2_{L^2(\mathrm{dist}_z^\alpha, \mathcal{N}_T)},$$

where $\mathcal{N}_T$ is the patch of T.

Proof (Ingredients)

- If $z \notin T$, the volume and jump terms are controlled via usual bubble function arguments.
- If $z \in T$:
  - The term $h_T^{\alpha+2-d}|\mathbf{F}|^2 \#(T \cap \{z\})$ is controlled via a function $\eta \in W_0^{1,\infty}(\Omega)$ such that

$$\eta(z) = 1, \quad \mathrm{supp}(\eta) \subset \mathcal{N}_T, \quad \|\nabla^k \eta\|_{L^\infty(\Omega)} = h_T^{-k}, \ k = 0, 1,$$

    and testing the error equation with $(\mathbf{F}\eta, 0)$.

📘 A. Allendes, E. Otárola, AJS. CMAME 2019.

Local efficiency II

Proof (continued)

- If $z \in T$[book]:
  - The volume term uses a bubble function φ_T such that $0 \le \varphi_T \le 1$ and
  $$\varphi_T(z) = 0, \quad |T| \lesssim \int_T \varphi_T, \quad |\varphi_T| \lesssim h_T^{-1}$$
  and $\operatorname{supp} \varphi_T \subset \overline{T^\star}$, where $T^\star$ is a subsimplex of T.
  Test the error equation with $(\varphi_T(\Delta \mathbf{u}_{\mathscr{T}} - \nabla p_{\mathscr{T}}), 0)$
  - The jump term uses a bubble function φ_S such that $0 \le \varphi_S \le 1$ and
  $$\varphi_S(z) = 0, \quad |S| \lesssim \int_S \varphi_S, \quad |\nabla \varphi_S| \lesssim h_T^{-1/2} |S|^{1/2}$$
  and $\operatorname{supp} \varphi_S \subset \overline{T_1^\star \cup T_2^\star}$, where $T_i^\star$ are subsimplices of T_i with $S = \bar{T}_1 \cap \bar{T}_2$.
  Test the error equation with $(\varphi_S [\![(\nabla \mathbf{u}_{\mathscr{T}} - p_{\mathscr{T}} \mathbb{I}) \cdot \mathbf{n}]\!], 0)$

□

[book] J.P. Agnelli, E. Garau, P. Morin. M2AN, 2014.

Local efficiency III

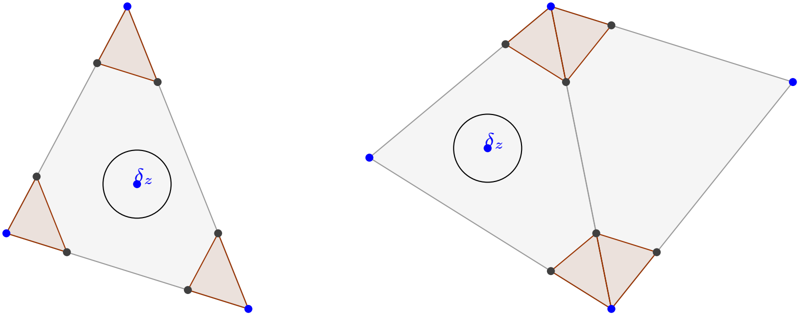

Figure: Support the bubble functions η_T, φ_T and φ_S.

Outline

Motivation

Analysis

Approximation

Conclusions and open problems

Conclusions

- We allow non-standard behavior, either in the forcing or constitutive law by considering weighted spaces.
- Stability of the Stokes projection on weighted spaces.
- A priori and a posteriori error analysis for linear and some nonlinear[a] models.
- Other models: Bousinessq[b], . . .

[a] A. Allendes, E. Otárola, AJS, SISC 2020.
[b] A. Allendes, E. Otárola, AJS, M3AN 2021.

Open questions

Analysis

- Navier Stokes for $d = 3$? It would require $q \neq 2$.
- Other models?

Approximation

- Stability of the Stokes projection:
  - Non quasi-uniform meshes?
  - Non convex domains?
  - $\varpi \notin A_{q/2}$?
- Error analysis for other models?
- Pseudo norm estimates for Smagorinsky?[📚]

[📚] J.W. Barrett, W.B. Liu, Math. Comp. 1993.

Thank You!

Well-posedness in Lipschitz domains I

- *Gårding–like inequality*: If $(\mathsf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\varpi, \Omega) \times L^2(\varpi, \Omega)/\mathbb{R}$ is a solution, then we have

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\varpi,\Omega)} + \|\mathsf{p}\|_{L^2(\varpi,\Omega)/\mathbb{R}} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\varpi^{-1},\Omega)'} + \|\mathsf{u}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathsf{p}\|_{H^{-1}(\mathcal{G})}.$$

  - Introduce a partition $\psi_i, \psi_\partial \in C_0^\infty(\Omega)$, $\psi_i + \psi_\partial \equiv 1$ with $\psi_i \equiv 1$ near $\Omega \setminus \mathcal{G}$ and $\psi_i \equiv 0$ near $\partial\Omega$. $\Omega_i = \operatorname{supp} \psi_i$ is C^1.
  - $\mathsf{u}_i = \mathsf{u}\psi_i$ and $\mathsf{p}_i = \mathsf{p}\psi_i$ are solutions on Ω_i, a C^1 domain, $\Longrightarrow$ use the weighted result for C^1 domains.
  - $\mathsf{u}_\partial = \mathsf{u}\psi_\partial$ and $\mathsf{p}_\partial = \mathsf{p}\psi_\partial$ are solutions on $\mathcal{G}$, a Lipschitz domain, $\Longrightarrow$ use the unweighted result for Lipschitz domains.

Well-posedness in Lipschitz domains II

- *Uniqueness*: From this it follows that, if $\mathbf{f} \equiv \mathbf{0}$, then $\mathsf{u} \equiv \mathbf{0}$ and $\mathsf{p} \equiv 0$.
- *A priori estimate*: Using the usual ADN contradiction argument we get that, if $(\mathsf{u}, \mathsf{p}) \in \mathbf{H}_0^1(\varpi, \Omega) \times L^2(\varpi, \Omega)/\mathbb{R}$ is a solution,

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\varpi,\Omega)} + \|\mathsf{p}\|_{L^2(\varpi,\Omega)/\mathbb{R}} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\varpi^{-1},\Omega)'}.$$

- *Existence*: By approximation, $(\mathsf{u}_k, \mathsf{p}_k) \in \mathbf{H}_0^1(\varpi, \Omega) \times L^2(\varpi, \Omega)/\mathbb{R}$ is a solution for $\mathbf{f}_k \in \mathbf{C}_0^\infty(\Omega)$, such that $\mathbf{f}_k \to \mathbf{f}$ in $\mathbf{H}_0^1(\varpi^{-1}, \Omega)'$. The a priori estimates allow us to pass to the limit.

Back

Well-posedness in convex polyhedra I

- Let $z \in \Omega$ and Q a cube centered in z. We decompose

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2, \quad \mathbf{f}_1 = \mathbf{f}\chi_{2Q}, \qquad g = g_1 + g_2, \quad \operatorname{supp} g_1 = 2Q.$$

  The decomposition of g is a Bogovskiĭ decomposition, i.e., it preserves the zero averages.

- $(\mathsf{u}^i, \mathsf{p}^i)$ solves the Stokes problem with data $(-\nabla\cdot\mathbf{f}_i, g_i)$.
- We estimate the oscillation of u, i.e., $\mathcal{M}^\sharp_\Omega\left[\nabla\mathsf{u}\right](z)$

$$\begin{aligned}\mathcal{M}^\sharp_\Omega\left[\nabla\mathsf{u}\right](z) &\approx ⨍_Q |\nabla\mathsf{u}(x) - \nabla\mathsf{u}^2(z)|\,\mathrm{d}x \\ &\leq ⨍_Q |\nabla\mathsf{u}^1(x)|\,\mathrm{d}x + ⨍_Q |\nabla\mathsf{u}^2(x) - \nabla\mathsf{u}^2(z)|\,\mathrm{d}x = N + F.\end{aligned}$$

Well-posedness in convex polyhedra II

- For N the data is supported on a cube. Since Ω is a convex polyhedron, for $s > 1$[📕],

$$N \lesssim \frac{1}{|Q|^{1/s}} \|\nabla \mathsf{u}^1\|_{\mathbf{L}^s(\Omega)} \lesssim \frac{1}{|Q|^{1/s}} \left(\|\mathbf{f}_1\|_{\mathbf{L}^s(2Q)} + \|g\|_{L^s(2Q)} \right)$$
$$\lesssim \mathcal{M}\left[|\mathbf{f}|^s\right](z)^{1/s} + \mathcal{M}\left[|g|^s(z)^{1/s}.$$

- For F we use the mixed derivative estimates

$$F \leq \frac{\ell(Q)^\sigma}{|Q|} \int_Q \int_{2Q^c} \frac{|\mathbf{f}(y)| + |g_2(y)|}{|z-y|^{3+\sigma}} \, \mathrm{d}y \, \mathrm{d}x \lesssim \mathcal{M}\left[|\mathbf{f}|\right](z) + \mathcal{M}\left[|g|\right](z).$$

- In conclusion

$$\mathcal{M}_\Omega^\sharp \left[\nabla \mathsf{u}\right](z) \lesssim \mathcal{M}\left[|\mathbf{f}|^s\right](z)^{1/s} + \mathcal{M}\left[|g|^s\right](z)^{1/s}.$$

- By simple scaling

$$\int_\Omega \varpi \left(\frac{1}{|\Omega|} \int_\Omega \nabla \mathsf{u} \, \mathrm{d}x \right)^q \lesssim \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

[📕] V. Maz'ya, J. Rossman, Math. Nachr. 2007.

Well-posedness in convex polyhedra III

- The weighted Fefferman-Stein inequality[1] implies

$$\left\| \nabla \mathsf{u} - \frac{1}{|\Omega|} \int_\Omega \nabla \mathsf{u} \, \mathrm{d}x \right\|_{\mathbf{L}^q(\varpi,\Omega)} \lesssim \| \mathcal{M}^\sharp_\Omega [\nabla \mathsf{u}] \|_{\mathbf{L}^q(\varpi,\Omega)}$$
$$\lesssim \left\| \mathcal{M} [|\mathbf{f}|^s]^{1/s} \right\|_{\mathbf{L}^q(\varpi,\Omega)} + \left\| \mathcal{M} [|g|^s]^{1/s} \right\|_{L^q(\varpi,\Omega)}.$$

- The continuity of the maximal function on weighted spaces finally gives

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^q(\varpi,\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

- The properties of the Bogovskiĭ operator on weighted spaces[2] imply

$$\|\mathsf{p}\|_{L^q(\varpi,\Omega)} \lesssim \sup_{\mathbf{v} \in \mathbf{W}^{1,q'}_0(\varpi',\Omega)} \frac{\int_\Omega \mathsf{p} \nabla \cdot \mathbf{v} \, \mathrm{d}x}{\|\nabla \mathbf{v}\|_{\mathbf{L}^{q'}(\varpi',\Omega)}}$$

meaning

$$\|\mathsf{p}\|_{L^q(\varpi,\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{L}^q(\varpi,\Omega)} + \|g\|_{L^q(\varpi,\Omega)}.$$

Back

[1] L. Diening, M. Růžička, K. Schumacher

[2] Acosta and Durán, 2017.

Stability of the Stokes projection I

- Approximate Dirac delta: $z \in T \in \mathscr{T}$, then $\tilde{\delta}_z \in C_0^\infty(T)$ with

$$\int_\Omega \tilde{\delta}_z \, \mathrm{d}x = 1, \quad \|\tilde{\delta}_z\|_{L^\infty(\Omega)} \lesssim h_T^{-d}, \quad \int_\Omega \tilde{\delta}_z \mathbf{v}_\mathscr{T} \, \mathrm{d}x = \mathbf{v}_\mathscr{T}(z), \; \forall \mathbf{v}_\mathscr{T} \in \mathcal{V}(\mathscr{T}).$$

- The regularized (derivative of the) Green's function:

$$-\Delta \tilde{\mathbf{G}} + \nabla \tilde{\lambda} = -\partial_i \tilde{\delta}_z \mathbf{e}_j.$$

- The pair $(\mathbf{G}_\mathscr{T}, \lambda_\mathscr{T}) \in \mathcal{V}(\mathscr{T}) \times \mathcal{P}(\mathscr{T})$ is its Galerkin approximation.
- Recall that[1], there is $\lambda \in (0,1)$,

$$\sup_{z \in \Omega} \|\sigma_y^{\mu/2} \nabla (\tilde{\mathbf{G}} - \mathbf{G}_\mathscr{T})\|_{\mathbf{L}^2(\Omega)} \lesssim h^{\lambda/2}, \quad \mu = d + \lambda$$

where the regularized distance[2] is

$$\sigma_y(x) = \left(|x-y|^2 + (\kappa h_\mathscr{T})^2 \right)^{1/2}$$

[1] V. Girault, R.H. Nochetto, R. Scott, Num. Math. 2015.

[2] F. Natterer, 1976. J.Nitchse, 1977. . . .

Stability of the Stokes projection II

- We have

$$
\begin{aligned}
a(\mathsf{u}, \tilde{\mathbf{G}}) + b_-(\mathsf{u}, \tilde{\lambda}) &= \int_\Omega \tilde{\delta}_z \partial_i \mathsf{u}^j \,\mathrm{d}x \\
a(\mathbf{u}_{\mathscr{T}}, \mathbf{G}_{\mathscr{T}}) + b_-(\mathbf{u}_{\mathscr{T}}, \lambda_{\mathscr{T}}) &= \partial_i \mathbf{u}^j_{\mathscr{T}}(z) \\
a(\mathbf{u}_{\mathscr{T}}, \tilde{\mathbf{G}} - \mathbf{G}_{\mathscr{T}}) + b_-(\mathbf{u}_{\mathscr{T}}, \tilde{\lambda} - \lambda_{\mathscr{T}}) &= 0 \\
a(\mathsf{u} - \mathbf{u}_{\mathscr{T}}, \mathbf{G}_{\mathscr{T}}) + b_-(\mathbf{G}_{\mathscr{T}}, \mathsf{p} - p_{\mathscr{T}}) &= 0.
\end{aligned}
$$

- Using that u and $\tilde{\mathbf{G}}$ are solenoidal, that $\mathbf{u}_{\mathscr{T}}$ and $\mathbf{G}_{\mathscr{T}}$ are discretely solenoidal, and that a is symmetric we eventually reach

$$
\partial_i \mathbf{u}^j_{\mathscr{T}}(z) = a(\mathsf{u}, \mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}) + b_-(\mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}, \mathsf{p}) + \int_\Omega \tilde{\delta}_z \partial_i \mathsf{u}^j \,\mathrm{d}x
$$

- Thus, with $\mathbf{E} = \mathbf{G}_{\mathscr{T}} - \tilde{\mathbf{G}}$

$$
\begin{aligned}
\int_\Omega \varpi |\partial_i \mathbf{u}^j_{\mathscr{T}}|^2 \,\mathrm{d}x &\lesssim \int_\Omega \varpi \left[\int_\Omega \nabla \mathsf{u} : \nabla \mathbf{E} \,\mathrm{d}x \right]^2 \mathrm{d}z + \int_\Omega \varpi \left[\int_\Omega \mathsf{p} \nabla\cdot \mathbf{E} \,\mathrm{d}x \right]^2 \mathrm{d}z \\
&+ \int_\Omega \varpi \left[\frac{1}{|T|} \int_T \partial_i \mathsf{u}^j \,\mathrm{d}x \right]^2 \mathrm{d}z \\
&= I + II + III.
\end{aligned}
$$

Stability of the Stokes projection III

- By continuity of the maximal function on weighted spaces

$$III \lesssim \int_\Omega \varpi \left| \mathcal{M}\left[\partial_i \mathsf{u}^j\right] \right|^2 \,\mathrm{d}z \lesssim \|\partial_i \mathsf{u}^j\|^2_{L^2(\varpi,\Omega)}.$$

- Using the regularized distance

$$I + II \lesssim \int_\Omega \varpi \left(\int_\Omega \sigma_z^{d+\lambda} |\nabla \mathbf{E}|^2 \,\mathrm{d}x \right) \left(\int_\Omega \frac{|\nabla \mathsf{u}|^2 + |\mathsf{p}|^2}{\sigma_z^{d+\lambda}} \,\mathrm{d}x \right) \mathrm{d}z$$

- We saw that

$$\int_\Omega \sigma_z^{d+\lambda} |\nabla \mathbf{E}|^2 \,\mathrm{d}x \lesssim h_{\mathscr{T}}^\lambda$$

- A dyadic decomposition shows that

$$h_{\mathscr{T}}^\lambda \int_\Omega \frac{\varpi(z)}{\sigma_z^{d+\lambda}(x)} \,\mathrm{d}z \lesssim \mathcal{M}[\varpi](x) \lesssim \varpi(x)$$

  where the last step requires $\varpi \in A_1$.

- In conclusion

$$I + II \lesssim \int_\Omega \varpi \left(|\nabla \mathsf{u}|^2 + |\mathsf{p}|^2 \right) \mathrm{d}x.$$