Dynamics of visco-elastic bodies with a cohesive interface

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based on joint works with R. Scala

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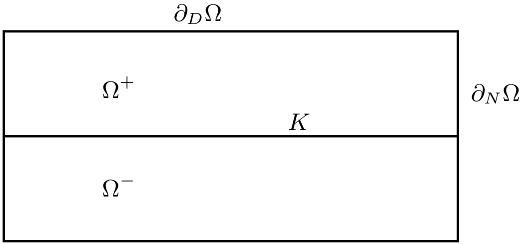
Bulk energies



Antiplane displacement u

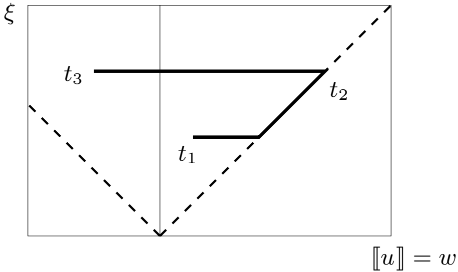
$$\mathcal{E}(u) = \tfrac{1}{2} \int_\Omega \mu |\nabla u|^2 \, dx \qquad \mathcal{K}(v) = \tfrac{1}{2} \int_\Omega \rho |v|^2 \, dx \qquad \mathcal{R}(v) = \tfrac{1}{2} \int_\Omega \eta |\nabla v|^2 \, dx$$

Internal (history) variable "$\xi(t) = \max_{[0,t]} |[\![u]\!]|$" on the interface (crack path) K

KKT

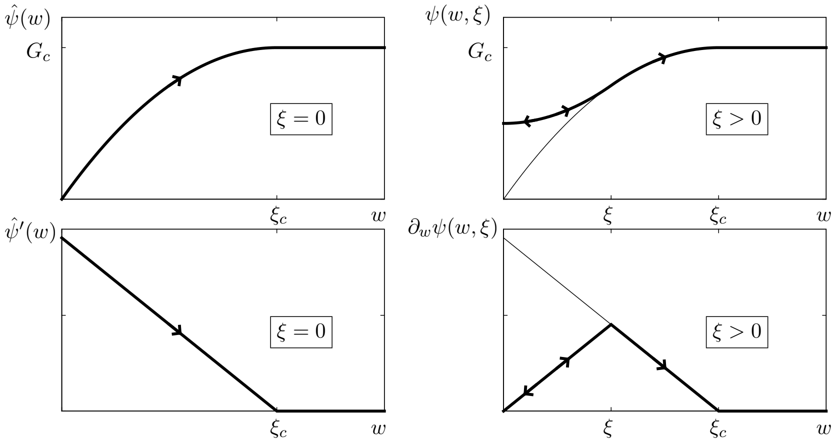
Karush-Kuhn-Tucker conditions

$$\dot{\xi}(t) \geq 0 \quad \text{and} \quad \dot{\xi}(t)(\xi(t) - |[\![u(t)]\!]|) = 0 \quad \text{and} \quad |[\![u(t)]\!]| \leq \xi(t)$$



horizontal (reversible) + diagonal (irreversible) directions inside the cone

Cohesive energy: loading and unloading



(keywords: finite tension, softening)

[Dagdale, Barenblatt, ..., Ortiz-Pandolfi]

$$\Psi(u,\xi) = \int_K \psi([\![u]\!],\xi)\, dr \quad \psi(w,\xi) = \begin{cases} \hat\psi(|w|) & \text{if } |w| \geq \xi \\ \hat\psi(\xi) - \frac{1}{2}\left(\frac{\hat\psi'(\xi)}{\xi}\right)(\xi^2 - w^2) & \text{if } |w| < \xi \end{cases}$$

Barenblatt

[Barenblatt]

those comparatively rare materials that are perfectly brittle. Experimental investigations show that when cracks appear some materials, which behave as highly plastic bodies in common tensile tests, fracture in such a way that plastic deformations, though present, are concentrated in a thin layer near the crack surface.

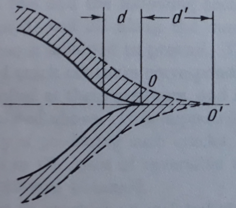


FIG. 16.

D. K. Felbeck and E. O. Orowan [28] carried out experiments on fracture of low-carbon steel plates with a saw-cut crack under conditions corresponding to Griffith's scheme of uniform extension. Experimental results are in good agreement with Griffith's formula, but the surface-energy density exceeds by about three orders of magnitude the surface tension of the material inves-

$d = \xi_c$ "cohesive zone"

$\xi_c \to 0^+$ brittle fracture (Griffith)

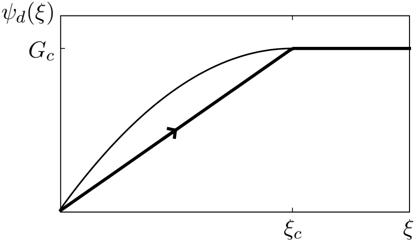
Issue: brittle fracture is incompatible with visco-elasticity (in q.s.)

[Dal Maso-Racca, Caponi]

Dissipated energy

For $\psi_d(\xi) = \psi(0, \xi)$

$$\Psi(u, \xi) = \int_K \psi_s(|[\![u]\!]|, \xi) + \psi_d(\xi) \, dr = \Psi_s(u, \xi) + \Psi_d(\xi)$$



ψ_d is monotone and bounded

Potential energy $\mathcal{F}(t, u, \xi) = \mathcal{E}(u) + \Psi(u, \xi) - \langle f(t), u \rangle$

Weak solutions

Let $U = \{u \in H^1 : u = 0 \text{ on } \partial_D \Omega\}$ with H^1-norm

Weak solutions in $W^{1,2}(0,T;U) \cap W^{2,2}(0,T;U^*)$

System of PDEs

$$\begin{cases} \rho \ddot{u}(t) + \partial_u \mathcal{F}(t, u(t), \xi(t)) + \partial_v \mathcal{R}(\dot{u}) \ni 0 \\ \dot{\xi}(t) \geq 0 \;\text{ and }\; \dot{\xi}(t)(\xi(t) - |[\![u(t)]\!]|) = 0 \;\text{ and }\; |[\![u(t)]\!]| \leq \xi(t) \\ u(0) = u_0,\; \xi(0) = \xi_0,\; \dot{u}(0) = v_0 \end{cases}$$

$$(\rho \ddot{u}(t), \phi)_{\mathcal{U}} + \partial_u \mathcal{F}(t, u(t), \xi(t); \phi) + \partial_v \mathcal{R}(\dot{u}(t))[\phi] \geq 0 \;\text{ for } \phi \in U$$

$$\partial_u \Psi(u, \xi; \phi) = \int_K \partial_w \psi([\![u]\!], \xi; \phi)\, dr$$

Energy identity

For every time t

$$\mathcal{F}(t, u(t), \xi(t)) + \mathcal{K}(\dot{u}(t)) = \mathcal{F}(0, u_0, \xi_0) + \mathcal{K}(v_0) + \int_0^t \partial_t \mathcal{F}(s, u(s), \xi(s))\, ds - \int_0^t \partial_v \mathcal{R}(\dot{u}(s))[\dot{u}(s)]\, ds$$

$$\begin{aligned} \mathcal{E}(u(t)) + \Psi_s(u(t), \xi(t)) + \mathcal{K}(\dot{u}(t)) = {} & \mathcal{E}(u_0) + \Psi_s(u_0, \xi_0) + \mathcal{K}(v_0) \\ & + \int_0^t \mathcal{P}_{ext}(s, u(s))\, ds \\ & - \int_0^t \partial_v \mathcal{R}(\dot{u}(s))[\dot{u}(s)]\, ds - \int_0^t \partial_\xi \Psi_d(\xi(s); \dot{\xi}(s))\, ds \end{aligned}$$

Adhesive energy and dissipation: $0 \leq z \leq 1$ and $\dot z \leq 0$

[Roubíček, Rossi-Roubíček, Scala, Thomas-Zanini]

$$\Phi_s(u,z) = \int_K z\,|[\![u]\!]|^2\,dr \qquad \Phi_d(z) = \int_K 1 - z\,dr$$

$$\Psi_s(u,\xi) = \int_K \Big(\frac{\hat\psi'(\xi)}{\xi}\Big)|[\![u]\!]|^2\,dr \qquad \Psi_d(\xi) = \int_K \psi_d(\xi)\,dr$$

$$\begin{cases} \rho\ddot u(t) + \partial_u\mathcal{F}(t,u(t),\xi(t)) + \partial_v\mathcal{R}(\dot u) \ni 0 \\ \xi(t) \in \operatorname{argmin}\{\mathcal{F}(t,u(t),\zeta) : \zeta \geq \xi(t)\} \\ \mathcal{F}(t,u(t),\xi(t)) + \mathcal{K}(\dot u(t)) \ \leq \text{ or } = \ \mathcal{F}(0,u_0,\xi_0) + \mathcal{K}(v_0) \\ \qquad\qquad + \int_0^t \partial_t\mathcal{F}(s,u(s),\xi(s))\,ds - \int_0^t \partial_v\mathcal{R}(\dot u(s))[\dot u(s)]\,ds \end{cases}$$

Strong solutions

For $\sigma(t) = \mu \nabla u(t) + \nu \nabla \dot{u}(t)$ the visco-elastic stress

Solutions in $W^{1,\infty}(0,T;U) \cap W^{2,\infty}(0,T;L^2)$ solve

$$\begin{cases} \rho \ddot{u}(t) = \operatorname{div} \sigma(t) + f(t) & \text{in } \Omega \\ \sigma(t)\, \nu = 0 & \text{in } \partial_N \Omega \\ \sigma^+(t)\, \nu = \sigma^-(t)\, \nu \in \partial_w \psi\big([\![u(t)]\!], \xi(t) \big) & \text{in } K \end{cases}$$

Compatibility assumptions on the initial conditions:

- $v_0 \in U$ with $[\![v_0]\!] = 0$ if $\xi_0 = 0$ and $\displaystyle\int_{\xi_0 > 0} \Big(\frac{\hat{\psi}'(\xi_0)}{\xi_0} \Big) |[\![v_0]\!]|^2 \, dr < \infty$

  (e.g. if there exists $\tau_0 > 0$ s.t. $|[\![u_0 + \tau_0 v_0]\!]| \le \xi_0$)

- there exists $w_0 \in L^2$ s.t. $\rho w_0 + \partial_u \mathcal{F}(0, u_0, \xi_0) + \partial_v \mathcal{R}(v_0) \ni 0$

Existence of weak solutions

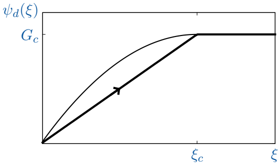
1) Replace ξ_0 with $\xi_\varepsilon = \max\{\xi_0, \varepsilon\}$ ⇨ "adhesive" regularization of Ψ

2) Time discretization $t_{n,k} = k\tau_n$ for $\tau_n = T/n$

$$\begin{cases} u_{n,k} \in \operatorname{argmin}\{\mathcal{J}(t_{n,k}, u, \xi_{n,k-1}) : u \in \mathcal{U}\} \\ \xi_{n,k} = \max\{\xi_{n,k-1}, |[\![u_{n,k}]\!]|\} \end{cases}$$

$$\mathcal{J} = \tfrac{1}{2}\rho \left\| \frac{u - 2u_{n,k-1} + u_{n,k-2}}{\tau_n} \right\|^2 + \mathcal{R}\left(\frac{u - u_{n,k-1}}{\tau_n}\right) + \mathcal{F}(t_{n,k}, u, \xi_{n,k-1})$$

$$\begin{cases} u_{n,k} \in \operatorname{argmin}\{\mathcal{J}(t_{n,k}, u, \xi_{n,k}) : u \in \mathcal{U}\} \\ \xi_{n,k} \in \operatorname{argmin}\{\mathcal{J}(t_{n,k}, u_{n,k}, \xi) : \xi \geq \xi_{n,k-1}\} \end{cases}$$



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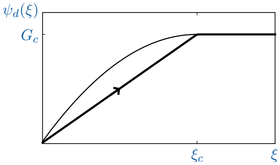
$$\begin{cases} \langle \rho \dot{v}_{n,k}, \phi \rangle + \partial_v \mathcal{R}(\dot{u}_{n,k})[\phi] + \partial_u \mathcal{F}(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi] = 0 \\ |[\![u_{n,k}]\!]| \le \xi_{n,k} \quad \dot{\xi}_{n,k}(|[\![u_{n,k}]\!]| - \xi_{n,k}) = 0 \quad \dot{\xi}_{n,k} \ge 0 \end{cases}$$



3) Piecewise constant/affine interpolation and ε-uniform compactness ...

$$\boxed{\|u_n\|_{W^{1,2}(0,T;U)} + \|\dot{u}_n\|_{W^{1,2}(0,T;U^*)} \le C} \quad \Rightarrow \quad \|\xi_n\|_{W^{1,2}(0,T;L^2)} \le C$$

by the discrete E-L equation $\partial_u \mathcal{J}(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi]$ with $\phi = u_{n,k} - u_{n,k-1}$

4) Improved convergence $\boxed{\xi_n \to \xi^\varepsilon \text{ in } L^2(0,T;L^2)}$ by KKT [N.-Vitali]

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5) Euler-Lagrange eq. $\partial_u \mathcal{J}(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi] = 0$ converge to

$$\rho(\ddot{u}^\varepsilon(t), \phi) + \partial_u \mathcal{F}(t, u^\varepsilon(t), \xi^\varepsilon(t))[\phi] + \partial_v \mathcal{R}(\dot{u}^\varepsilon)[\phi] = 0$$

6) KKT from discrete to continuum ...

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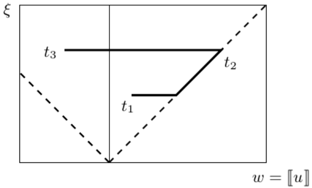
7) Convergence for $\varepsilon \to 0$ ...

$$\rho(\ddot{u}(t), \phi) + \partial_u \mathcal{F}(t, u(t), \xi(t))[\phi] + \partial_v \mathcal{R}(\dot{u})[\phi] \ni 0$$

Energy identity

1) $t \mapsto \mathcal{K}(\dot u(t)) + \mathcal{E}(u(t)) + \Psi(u(t),\xi(t))$ is AC



$$|\Psi(u(t),\xi(t)) - \Psi(u_0,\xi_0)| \le C \int_0^t \|[\![\dot u(s)]\!]\|\, ds \qquad \text{by KKT}$$

2) $\dot{\mathcal{K}} + \dot{\mathcal{E}} + \dot\Psi = (\rho \ddot u(t), \dot u(t)) + \partial_u \mathcal{E}(u(t))[\dot u(t)] + \partial_u \Psi(u(t),\xi(t);\dot u(t))$

$$\Psi(u(t),\xi(t)) - \Psi(u_0,\xi_0) = \int_0^t \partial_u \Psi\big(u(s),\xi(s);\dot u(s)\big)\, ds \qquad \text{by KKT}$$

3) E-L equation for $\phi = \dot u$

$$(\rho \ddot u(t), \dot u(t)) + \partial_u \mathcal{F}(t,u(t),\xi(t);\dot u(t)) + \partial_v \mathcal{R}(\dot u(t))[\dot u(t)] = 0.$$

equality relies on $[\![\dot u(t)]\!] = 0$ if $\xi(t) = 0$ by KKT

Existence of strong solutions

1) Let $\bar{\xi}_\varepsilon = \max\{\xi_0, \varepsilon\}$. Replace the i.c. u_0 and ξ_0 with

$$u_\varepsilon \in \operatorname{argmin} \left\{ \mathcal{F}(0, u, \bar{\xi}_\varepsilon\}) + \langle \eta \nabla v_0, \nabla u \rangle + \langle \rho w_0, u \rangle : u \in U \right\}$$
$$\xi_\varepsilon = \max\{\bar{\xi}_\varepsilon, |[\![u_\varepsilon]\!]|\}$$

Then $u_\varepsilon \to u_0$ and $\xi_\varepsilon \to \xi_0$ (by Γ-convergence)

the compatibility assumptions hold

2) Time discretization and incremental scheme ...

3) Interpolation and (tricky) ε-uniform compactness for the speed v_n

$$\boxed{\|v_n\|_{L^\infty(0,T;U)} + \|\dot{v}_n\|_{L^\infty(0,T;L^2)} \leq C.}$$

by studying the difference of the E-L equations

$$\Big(\partial_u \mathcal{J}(t_{n,k}, u_{n,k}, \xi_{n,k}) - \partial_u \mathcal{J}(t_{n,k-1}, u_{n,k-1}, \xi_{n,k-1}) \Big) [v_{n,k} - v_{n,k-1}] = 0$$

Existence of strong solutions

Linear (bulk) terms are easy but

$$\partial_u \Psi(u_{n,k}, \xi_{n,k})[v_{n,k} - v_{n,k-1}] - \partial_u \Psi(u_{n,k-1}, \xi_{n,k-1})[v_{n,k} - v_{n,k-1}] = \\ = \int_K \tau_n \alpha_{n,k} [\![v_{n,k}]\!] [\![v_{n,k} - v_{n,k-1}]\!] \, ds$$

where $\tau_n v_{n,k} = u_{n,k} - u_{n,k-1}$ and

$$\alpha_{n,k} = \frac{c_{n,k} [\![u_{n,k}]\!] - c_{n,k-1} [\![u_{n,k-1}]\!]}{[\![u_{n,k} - u_{n,k-1}]\!]} \quad \text{for} \quad c_{n,k} = \frac{\hat{\psi}'(\xi_{n,k})}{\xi_{n,k}}$$

Use the visco-elastic term

$$\| \nabla v_{n,k} - \nabla v_{n,k-1} \|^2_{L^2} \geq C \int_K [\![v_{n,k} - v_{n,k-1}]\!]^2 \, ds$$

After tricky estimates on $\alpha_{n,k} - \alpha_{n,k-1}$ ... Gronwall Lemma gives compactness

Generalizations

Conditions on the energy density $\hat{\psi} : [0, +\infty) \to [0, +\infty)$ for weak solutions

- $\hat{\psi}(0) = 0$, $\hat{\psi}(w) > 0$ for $w > 0$, $\hat{\psi}(w) = \hat{\psi}(\xi_c)$ for $w \geq \xi_c > 0$
- $\hat{\psi}$ is concave
- $\hat{\psi}$ is of class C^1 in $[0, +\infty)$ and of class C^2 in $[0, \xi_c]$.

Similarly for $\xi_c = +\infty$ ...

A non-linear dissipation pseudo-potential $\psi_d(\xi) = \hat{\psi}(\xi) - \frac{1}{2}\hat{\psi}'(\xi)\xi$

Further conditions for strong solutions

- $\hat{\psi}'$ is concave in $[0, \xi_c]$
- there exists $c > 0$ such that $\mu\|\nabla u\|^2 + \beta\,\|[\![u]\!]\|^2 \geq c\,\|\nabla u\|^2$
  where $\beta = \min\{\hat{\psi}''(w) : w \in [0, \xi_c]\} < 0$

Some open problems

Uniqueness ...

Vectorial setting: $u \in H^1(\Omega;\mathbb{R}^N)$ and $N = 2, 3$

$$\Psi(u,\xi) = \int_K \psi([\![u]\!],\xi)\, dr \qquad \hat\psi(w) = \begin{cases} \dots & \text{if } w\cdot\nu \ge 0 \\ +\infty & \text{otherwise} \end{cases}$$

Some results for the existence of weak solutions ... [Scala, Scala-Schimperna]

Quasi-static limit by time rescaling: $v_0 = 0$ and $\bar f_n(t) = f(t/n)$ for $t \in [0, nT]$

consider $\bar u_n : [0, nT] \to U$ and $u_n(t) = \bar u_n(nt) : [0,T] \to U$ solving

$$\rho_n \ddot u_n(t) + \eta_n \partial_v \mathcal{R}(\dot u_n(t)) + \partial_u \mathcal{F}(t, u_n(t), \xi_n(t)) \ni 0$$

where $\rho_n = \rho n^{-2}$ and $\eta_n = \eta n^{-1}$

Issue: characterize as $n \to \infty$ the limit q.s. evolution [Scala, Roubíček, Scilla-Solombrino]

Open problem: which q.s. evolution?

Qualitatively: discontinuous evolutions $u \in BV(0,T;H^1)$ and $\xi \in BV(0,T;L^2)$

- $\partial_u \mathcal{F}(t,u(t),\xi(t)) \ni 0$ and “KKT”
- characterization of the transition in discontinuities

$$[0,S] \ni s \mapsto \bar{u}(s) \ \text{ s.t. } \ \bar{u}(0) = u^-(t) \ \text{ and } \ \bar{u}(S) = u^+(t)$$

- energy identity

$$\begin{aligned}\mathcal{F}(t,u(t),\xi(t)) = \mathcal{F}(0,u_0,\xi_0) &+ \int_0^t \partial_t \mathcal{F}(s,u(s),\xi(s))\,ds \\ &+ \sum_{t_j \in J} [\![\mathcal{F}(t_j,u(t_j),\xi(t_j))]\!] \\ &+ \mu([0,t])\end{aligned}$$

[Scala, Roubíček]

A q.s. evolution with no inertia

Balanced (or vanishing) viscosity evolution for $\rho = 0$ and $\eta_n \to 0$

Discontinuous evolutions $u \in BV(0,T;H^1)$ and $\xi \in BV(0,T;L^2)$

Parametrization $s \mapsto (t(s), u(s), \xi(s))$ [Mielke-Efendiev, N., N.-Vitali]

- if $t(s)$ is a continuity time

$$\partial_u \mathcal{F}(t(s), u(s), \xi(s)) \ni 0$$

$$\xi'(s) \geq 0, \quad \xi'(s)(\xi(s) - |[\![u(s)]\!]|) = 0, \quad |[\![u(s)]\!]| \leq \xi(s)$$

- if $[s_1, s_2]$ is a parametrization of a jump then $t = t(s)$ and

$$\eta\, \partial_v \mathcal{R}(u'(s)) + \partial_u \mathcal{F}(t, u(s), \xi(s)) \ni 0$$

$\eta\, u_n'(s) \approx \dot{\boldsymbol{u}}_n(t_n(s))\, \eta_n$ where $\eta_n = \eta n^{-1}$