A ‘liquid-solid’ phase transition in a simple model for flocking

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Joint work with Elliott Lieb (Princeton)

Self-assembly/aggregation in biology and physics

Simple model for flocking/swarming/herding behavior in biology based on a competition between short-range repulsion and long-range attraction

Phase transition as the total number of birds, etc. changes

Think of tightly packed emperor penguins huddling in an Antarctic winter...

The model of Burchard–Choksi–Topaloglu

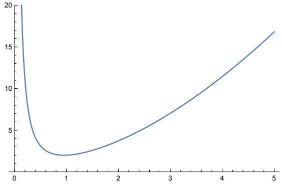
Fix parameters $0 < \lambda < N$ and $\alpha > 0$.

Energy of 'particle' configuration with density $\rho : \mathbb{R}^N \to [0,1]$

$$\mathcal{E}[\rho] = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho(x) \left(\frac{1}{|x-y|^\lambda} + |x-y|^\alpha \right) \rho(y)\, dx\, dy$$

Ground state energy (at total 'particle' number $m > 0$)

$$E(m) = \inf \left\{ \mathcal{E}[\rho] : \ 0 \leq \rho \leq 1 \,, \ \int_{\mathbb{R}^N} \rho(x)\, dx = m \right\}$$



Interaction kernel $k(|x-y|)$, here $k(r) = r^{-\lambda} + r^\alpha$
repulsive at short distances, attactive at long distances
$k(r) \to \infty$ as $r \to 0$ and $r \to \infty$, unique min in-between

Important feature: constraint $\rho \leq 1$ on maximal value of density

Goal: understand minimizers for $E(m)$. Today, mostly for $m \gg 1$.

The main result

$$E(m) = \inf \left\{ \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho(x) \left(\frac{1}{|x-y|^\lambda} + |x-y|^\alpha \right) \rho(y)\, dx\, dy :\ 0 \le \rho \le 1,\ \int_{\mathbb{R}^N} \rho\, dx = m \right\}$$

Theorem (F.–Lieb)

Let $0 < \lambda < N-1$ and $\alpha > 0$. Then there is an $m_* < \infty$ (depending on N, α and λ) such that for all $m > m_*$ the only minimizers for $E(m)$ are characteristic fcns of balls.

- It is not hard to see that minimizers for large m are close, in a suitable sense, to characteristic functions of balls. The theorem says they are exactly balls!
- The assumption $\lambda < N-1$ (as opposed to $\lambda < N$) is necessary. For $N-1 \le \lambda < N$, characteristic fcns of balls are not even critical points.
- Our proof is based on quantitative rearrangement inequalities, and fundamentally different from arguments by Burchard–Choksi–Topaloglu and Lopes.
- Open questions about long-time convergence and convergence rate for the corresponding Wasserstein gradient flow (Craig–Kim–Yao, Craig–Topaloglu, . . .)

Competition between attraction and repulsion

$$\mathcal{E}[\rho] = \mathcal{I}_{-\lambda}[\rho] + \mathcal{I}_\alpha[\rho]$$

with

$$\mathcal{I}_{-\lambda}[\rho] = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho(x)\,\rho(y)}{|x-y|^\lambda}\,dx\,dy\,, \quad \mathcal{I}_\alpha[\rho] = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)|x-y|^\alpha \rho(y)\,dx\,dy\,.$$

Rescale $\rho(x) = \sigma(x/m^{1/N})$ with $0 \le \sigma \le 1$, $\int_{\mathbb{R}^N} \sigma(x)\,dx = 1$,

$$\mathcal{E}[\rho] = m^{\frac{\alpha}{N}} \left(\mathcal{I}_\alpha[\sigma] + m^{-\frac{\lambda+\alpha}{N}} \mathcal{I}_{-\lambda}[\sigma] \right).$$

For large m, the λ-term is a small perturbation of the α-term.

- One can show that

  $$\inf\left\{ \mathcal{I}_\alpha[\sigma] :\ 0 \le \sigma \le 1, \int \sigma = 1 \right\} = \mathcal{I}_\alpha[\mathbb{1}_{\mathcal{B}}] \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = 1\,.$$

  The α-term wants ρ to be a ball (long range attraction).

- However, one can also show that

  $$\sup\left\{ \mathcal{I}_{-\lambda}[\sigma] :\ 0 \le \sigma \le 1, \int \sigma = 1 \right\} = \mathcal{I}_{-\lambda}[\mathbb{1}_{\mathcal{B}}] \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = 1\,.$$

  The λ-term wants ρ *not* to be a ball (short range repulsion).

- The two terms compete!

Main result: For large m, the α-term wins over the λ-term and there is no compromise.

The three key ingredients

$$A[\rho] = \inf\left\{ (2\|\rho\|_{L^1})^{-1} \, \|\rho - \mathbb{1}_{\mathcal{B}}\|_{L^1} : \text{ ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1} \right\},$$

$$A_{\mathrm{H}}[\rho] = \inf\left\{ \theta \in [0,1] : \ \mathbb{1}_{(1-\theta)\mathcal{B}} \le \rho \le \mathbb{1}_{(1+\theta)\mathcal{B}}\,, \ \text{ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1} \right\}.$$

Ingredient I

Let $\alpha > 0$. Then for all $\rho \in L^1(\mathbb{R}^N)$ with $0 \le \rho \le 1$,

$$\mathcal{I}_\alpha[\rho] - \mathcal{I}_\alpha[\mathbb{1}_{\mathcal{B}}] \ge c\, \|\rho\|_{L^1}^{2+\frac{\alpha}{N}} A[\rho]^2 \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1}.$$

Ingredient II

Let $0 < \lambda < N-1$. Then for all $\rho \in L^1(\mathbb{R}^N)$ with $0 \le \rho \le 1$,

$$\mathcal{I}_{-\lambda}[\rho] - \mathcal{I}_{-\lambda}[\mathbb{1}_{\mathcal{B}}] \ge -C\, \|\rho\|_{L^1}^{2-\frac{\lambda}{N}} A_{\mathrm{H}}[\rho]^2 \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1}.$$

Ingredient III

If ρ is a minimizer for $E(m)$ with $m \ge m_*$, then

$$A_{\mathrm{H}}[\rho] \le C\, A[\rho].$$

These propositions imply the main theorem since for all large m and a minimizer ρ,

$$0 \ge \mathcal{E}[\rho] - \mathcal{E}[\mathbb{1}_{\mathcal{B}}] \ge c \left(\|\rho\|_{L^1}^{2+\frac{\alpha}{N}} - C\|\rho\|_{L^1}^{2-\frac{\lambda}{N}} \right) A_{\mathrm{H}}[\rho]^2 \ge \tfrac{c}{2}\, \|\rho\|_{L^1}^{2+\frac{\alpha}{N}} A_{\mathrm{H}}[\rho]^2$$

Rearrangement

For a measurable set $E \subset \mathbb{R}^N$ of finite measure, let

$$E^* = \text{ball in } \mathbb{R}^N, \text{ centered at the origin, of measure } |E^*| = |E| \,.$$

Riesz rearrangement inequality: for all $E, F, G \subset \mathbb{R}^N$ of finite measure,

$$\iint_{E\times F} \mathbb{1}_G(x-y)\,dx\,dy \leq \iint_{E^*\times F^*} \mathbb{1}_{G^*}(x-y)\,dx\,dy$$

Cases of equality: Burchard (1996), stability bound: Christ (2017)
Here, we only need the special case $E = F$ and $G = b =$ ball centered at the origin:

$$\iint_{E\times E} \mathbb{1}_b(x-y)\,dx\,dy \leq \iint_{E^*\times E^*} \mathbb{1}_b(x-y)\,dx\,dy$$

(implies isoperimetric ineq!), but we allow $0 \leq \rho \leq 1$ with fixed measure instead of $\mathbb{1}_E$.

Theorem (Christ, F.–Lieb)

Let $0 < \delta \leq \frac{1}{2}$. Then there is a $c_{N,\delta} > 0$ such that for any $\rho \in L^1(\mathbb{R}^N)$ with $0 \leq \rho \leq 1$ and any ball $b \subset \mathbb{R}^N$ with

$$\delta \leq |b|^{1/N}/(2\|\rho\|_{L^1}^{1/N}) \leq 1-\delta$$

one has, with $\mathcal{B} =$ ball, concentric with b, of measure $|\mathcal{B}| = \|\rho\|_{L^1}$,

$$\frac{1}{2}\iint_{\mathcal{B}\times\mathcal{B}} \mathbb{1}_b(x-y)\,dx\,dy - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\mathbb{1}_b(x-y)\rho(y)\,dx\,dy \geq c_{N,\delta}\,\|\rho\|_{L^1}^2\, A[\rho]^2 \,.$$

Three consequences of the rearrangment theorem

The quantitative rearrangement theorem says

$$\boxed{\frac{1}{2}\iint_{\mathcal{B}\times\mathcal{B}} \mathbb{1}_b(x-y)\,dx\,dy - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\mathbb{1}_b(x-y)\rho(y)\,dx\,dy \geq c_{N,\delta}\,\|\rho\|_{L^1}^2\,A[\rho]^2}$$

provided that $\delta \leq |b|^{1/N}/(2\|\rho\|_{L^1}^{1/N}) \leq 1-\delta$.

First consequence of this is Key Ingredient I: For any $\alpha > 0$,

$$\boxed{\mathcal{I}_\alpha[\rho] - \mathcal{I}_\alpha[\mathbb{1}_\mathcal{B}] \geq c_{N,\alpha}\,\|\rho\|_{L^1}^{2+\frac{\alpha}{N}}\,A[\rho]^2} \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1}\,.$$

Proof. Use

$$|x|^\alpha = \alpha\int_0^\infty \left(1-\mathbb{1}_{b_r}(x)\right) r^{\alpha-1}\,dr$$

to write

$$\mathcal{I}_\alpha[\rho]-\mathcal{I}_\alpha[\mathbb{1}_\mathcal{B}] = \alpha\int_0^\infty \left(\frac{1}{2}\iint_{\mathcal{B}\times\mathcal{B}} \mathbb{1}_{b_r}(x-y)dxdy - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\mathbb{1}_{b_r}(x-y)\rho(y)dxdy\right) r^{\alpha-1}dr$$

The integrand is ≥ 0 (by Riesz and the bathtub principle) and $\geq c_{N,\delta}\,\|\rho\|_{L^1}^2\,A[\rho]^2$ for $r \sim \|\rho\|_{L^1}^{1/N}$ (by our rearrangement theorem). □

Three consequences of the rearrangment theorem. Cont'd

The quantitative rearrangement theorem says

$$\boxed{\frac{1}{2}\iint_{\mathcal{B}\times\mathcal{B}} \mathbb{1}_b(x-y)\,dx\,dy - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\mathbb{1}_b(x-y)\rho(y)\,dx\,dy \geq c_{N,\delta}\,\|\rho\|_{L^1}^2\,A[\rho]^2}$$

provided that $\delta \leq |b|^{1/N}/(2\|\rho\|_{L^1}^{1/N}) \leq 1-\delta$.

Second consequence (not needed here) is a quantitative rearrangement ineq for Riesz potentials: For any $0<\lambda<N$,

$$\boxed{\mathcal{I}_{-\lambda}[\mathbb{1}_{\mathcal{B}}] - \mathcal{I}_{-\lambda}[\rho] \geq c_{N,\lambda}\,\|\rho\|_{L^1}^{2-\frac{\lambda}{N}}\,A[\rho]^2} \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = \|\rho\|_{L^1}\,.$$

Due to Burchard–Chambers for $\lambda=1$, $N=3$, Fusco–Pratelli for $0<\lambda<N-1$.

Third consequence (not needed here) is a quantitative fractional isoperimetric ineq: For any $0<s<1$,

$$\boxed{\mathrm{Per}_s\,E - \mathrm{Per}_s\,\mathcal{B} \geq c_{N,s}\,|E|^{1-\frac{s}{N}}\,A[\mathbb{1}_E]^2} \qquad \text{with a ball } \mathcal{B} \text{ of measure } |\mathcal{B}| = |E|\,.$$

Weaker than Figalli–Fusco–Maggi–Millot–Morini, since we cannot let $s\to 1$. (Related to uncontrolled dependence of $c_{\delta,N}$ as $\delta\to 0$. Open problem!)

THANK YOU FOR YOUR ATTENTION!

https://tenor.com/view/penguins-bye-funny-animals-gif-14143308