Multiplicative inequalities on BMO

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Multiplicative inequality

For $1 \le p \le r < \infty$ there is a trivial multiplicative inequality:

$$\|\varphi\|_{L^r} \le \|\varphi\|_{L^p}^{p/r} \|\varphi\|_{L^\infty}^{1-p/r}$$

for $\varphi \in L^\infty \cap L^p$.

Analog for BMO:

$$\|\varphi\|_{L^r}^r \le C \|\varphi\|_{L^p}^p \|\varphi\|_{\mathrm{BMO}(\Omega)}^{r-p} \tag{1}$$

for $\varphi \in \mathrm{BMO}(\Omega) \cap L^p$, where C depends on p, r, Ω, and the choice of norm in BMO.

Chen–Zhu, 2005: for the case $\Omega = \mathbb{R}^d$ one has (1) with $C \le (C(d)r)^r$. Application to the Navier–Stokes equations (Kozono–Taniuchi, 2000).

We investigate constants C in (1) using Bellman function techniques.

BMO spaces

Let $\Omega \subset \mathbb{R}^d$ be a ball, a cube, or $\mathbb{R}^d$ itself. Let $Q \subset \Omega$ be a ball or a cube.

$$\langle \varphi \rangle_Q = \frac{1}{|Q|} \int_Q \varphi, \qquad \varphi \in L^1(Q).$$

Let $\varphi \in L^1_{\mathrm{loc}}(\Omega)$. Then, $\varphi \in \mathrm{BMO}(\Omega)$ if

$$\|\varphi\|^2_{\mathrm{BMO}(\Omega)} = \sup_{Q \subset \Omega} \langle |\varphi - \langle \varphi \rangle_Q|^2 \rangle_Q < +\infty.$$

One can use any $p \in [1, +\infty)$ instead of 2, all the $p-$norms are equivalent. But we will work with quadratic (semi-)norm.

Several cases:

- $\Omega = [0,1]$, Q are subsegments;
- $\Omega = \mathbb{T}$, Q are arcs;
- $\Omega = \mathbb{R}$, Q are segments;
- $\Omega = [0,1]^d$ or $\Omega = \mathbb{T}^d$, Q are subcubes;
- $\Omega = \mathbb{R}^d$, Q are cubes or balls;

Our results: sharp constants

$$\langle|\varphi|^r\rangle_\Omega \le C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|_{\mathrm{BMO}(\Omega)}^{r-p}, \qquad \langle\varphi\rangle_\Omega = 0; \tag{2}$$

$$\langle|\varphi|^r\rangle_\Omega \le C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|_{\mathrm{BMO}(\Omega)}^{r-p}. \tag{3}$$

Theorem 1

For $\Omega = [0,1]$ *the* *sharp* *constant in* (2) *is given by*:

$$C(p,r;[0,1]) = \frac{\Gamma(r+1)}{\Gamma(p+1)}, \qquad 1 \le p, \quad \max(2,p) \le r < +\infty; \tag{4}$$

$$C(p,r;[0,1]) = \ldots, \qquad 1 \le p \le r < 2. \tag{5}$$

Theorem 2

For $\Omega = \mathbb{T}$ *the* *sharp* *constant in* (2) *is the same*: $C(p,r;\mathbb{T}) = C(p,r;[0,1])$.

Theorem 3

For $\Omega = \mathbb{R}$ *the* *sharp* *constant in* (3) *is the same*: $C(p,r;\mathbb{R}) = C(p,r;[0,1])$.

Our results: estimates

$$\langle|\varphi|^r\rangle_\Omega \leq C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|_{\mathrm{BMO}(\Omega)}^{r-p}, \qquad \langle\varphi\rangle_\Omega = 0; \tag{2}$$

$$\langle|\varphi|^r\rangle_\Omega \leq C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|_{\mathrm{BMO}(\Omega)}^{r-p}. \tag{3}$$

Corollary 4

For $\Omega = [0,1]^d$ *or* $\Omega = \mathbb{T}^d$ *inequality* (2) *holds with*

$$C(p,r;\Omega) = \mathfrak{C}(d)^{r-p}C(p,r;[0,1]), \qquad \textit{where} \quad \mathfrak{C}(d) = \frac{1+2^d}{2^{1+d/2}}.$$

For $\Omega = \mathbb{R}^d$ *inequality* (3) *holds with the same constant.*

Monotonic rearrangement acts from $\mathrm{BMO}([0,1]^d)$ to $\mathrm{BMO}([0,1])$, its "norm" is bounded by $\mathfrak{C}(d)$.

Grasia-type norms on $\mathrm{BMO}(\mathbb{R}^d)$

For $y \in \mathbb{R}^d$ and $t > 0$ consider the Poisson kernel and the heat kernel:

$$P_t(y) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \frac{t}{(t^2 + |y|^2)^{\frac{d+1}{2}}}, \qquad H_t(y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|y|^2}{2t}}. \tag{6}$$

We will write K_t for both P_t and H_t.
For a function φ on $\mathbb{R}^d$ define its K-extension onto $\mathbb{R}^d \times \mathbb{R}_+$ by convolution:

$$\varphi_K(y,t) = (K_t * \varphi)(y), \qquad y \in \mathbb{R}^d, \ t > 0.$$

It is known that

$$\|\varphi\|_K := \sup_{(y,t)\in\mathbb{R}^d\times\mathbb{R}_+} \left(\varphi_K^2(y,t) - \varphi_K(y,t)^2\right)^{\frac{1}{2}}$$

is an equivalent norm on $\mathrm{BMO}(\mathbb{R}^d)$. For $K = P$ this norm is called the Garsia norm.

Our results: estimates via Garsia-type norms

Theorem 5

If $1 \le p \le r < \infty$, *then*

$$\|\varphi\|^r_{L^r(\mathbb{R}^d)} \le C(p, r; [0,1]) \cdot \|\varphi\|^p_{L^p(\mathbb{R}^d)} \cdot \|\varphi\|^{r-p}_K, \qquad \varphi \in L^p(\mathbb{R}^d),$$

where the kernel K *is either the Poisson kernel or the heat kernel.*

Slavin–Z.:

$$\|\varphi\|_H \le \tilde{C}\sqrt{d}\,\|\varphi\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

Corollary 6

If $1 \le p \le r < \infty$, *then*

$$\|\varphi\|^r_{L^r(\mathbb{R}^d)} \le C(p, r; [0,1]) \cdot (\tilde{C}\sqrt{d})^{r-p} \cdot \|\varphi\|^p_{L^p(\mathbb{R}^d)} \cdot \|\varphi\|^{r-p}_{\mathrm{BMO}(\mathbb{R}^d)}, \qquad \varphi \in L^p(\mathbb{R}^d).$$

The main Bellman function

Let $d = 1$, $I = [0,1]$.

$$\boldsymbol{B}_{p,r}(x_1,x_2,x_3) = \sup\Big\{\langle|\varphi|^r\rangle_I \colon \|\varphi\|_{\mathrm{BMO}(I)} \leq 1,\ \langle\varphi\rangle_I = x_1,\ \langle\varphi^2\rangle_I = x_2,\ \langle|\varphi|^p\rangle_I = x_3\Big\}.$$

Inequality (2) for $\Omega = I$ is equivalent to

$$\boldsymbol{B}_{p,r}(0,x_2,x_3) \leq C(p,r,[0,1]) \cdot x_3$$

for $(0,x_2,x_3)$ in the domain Ω of $\boldsymbol{B}_{p,r}$:

$$\Omega = \Big\{x \in \mathbb{R}^3 \colon x_1^2 \leq x_2 \leq x_1^2 + 1,\ \boldsymbol{B}_p^-(x_1,x_2) \leq x_3 \leq \boldsymbol{B}_p^+(x_1,x_2)\Big\},$$

where

$$\boldsymbol{B}_p^-(x_1,x_2) = \inf\Big\{\langle|\varphi|^p\rangle_I \colon \|\varphi\|_{\mathrm{BMO}(I)} \leq 1,\ \langle\varphi\rangle_I = x_1,\ \langle\varphi^2\rangle_I = x_2\Big\},$$

$$\boldsymbol{B}_p^+(x_1,x_2) = \sup\Big\{\langle|\varphi|^p\rangle_I \colon \|\varphi\|_{\mathrm{BMO}(I)} \leq 1,\ \langle\varphi\rangle_I = x_1,\ \langle\varphi^2\rangle_I = x_2\Big\}.$$

These two functions were studied by Slavin and Vasyunin in 2012.

Properties of $B_p^{\pm}$

Domain of $\boldsymbol{B}_p^{\pm}$ is

$$\omega = \{x \in \mathbb{R}^2 \colon x_1^2 \leq x_2 \leq x_1^2 + 1\}.$$

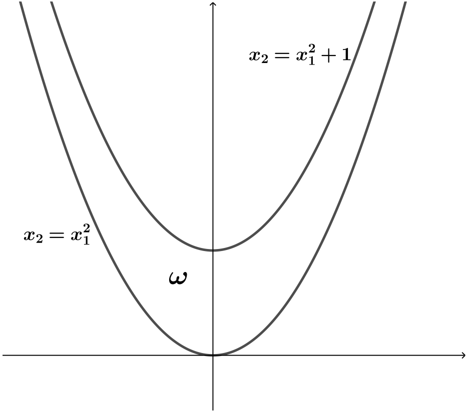
Boundary conditions:

$$\boldsymbol{B}_p^{\pm}(t, t^2) = |t|^p, \qquad t \in \mathbb{R}. \tag{7}$$

$\boldsymbol{B}_p^{+}$ is the (pointwise) minimal locally concave function on ω satisfying (7).

$\boldsymbol{B}_p^{-}$ is the (pointwise) maximal locally convex function on ω satisfying (7).

Local concavity (convexity) on ω means concavity (convexity) on any segment in ω.



Construction of A_{m_p}

Define

$$m_p(u) = p\int_u^{+\infty} e^{u-t}t^{p-1}dt, \quad u \geq 0.$$

For $u \geq 0$ and $x = (x_1, x_2) \in S_+(u)$:

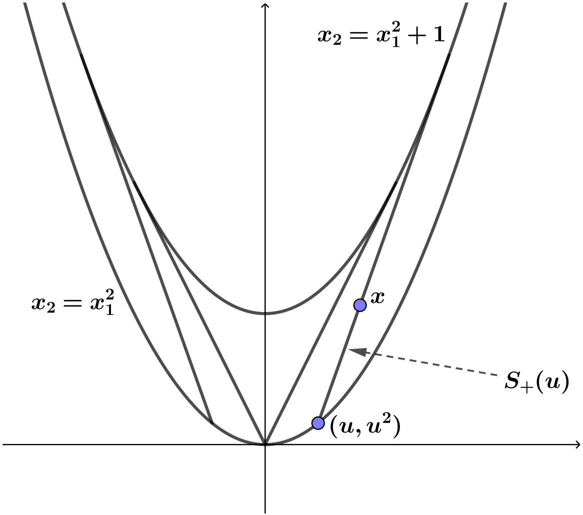
$$A_{m_p}(x) = u^p + m_p(u)(x_1 - u).$$

A_{m_p} is even w.r.t. x_1:

$$A_{m_p}(-x_1, x_2) = A_{m_p}(x_1, x_2), \quad x \in \omega.$$

In the "angle", $-1 \leq x_1 \leq 1$, $|x_1| \leq x_2 \leq x_1^2 + 1$:

$$A_{m_p}(x_1, x_2) = \frac{m_p(0)}{2}x_2.$$



Construction of A_{k_p}

Define

$$k_p(u) = p\int_1^u e^{t-u}t^{p-1}dt, \qquad u \geq 1.$$

For $u \geq 1$ and $x = (x_1, x_2) \in S_-(u)$:

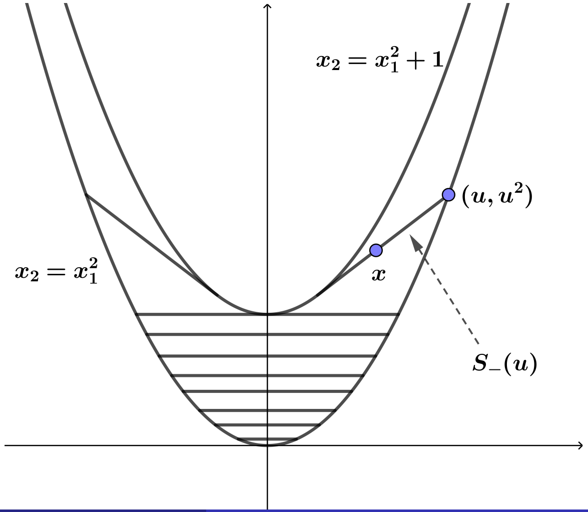
$$A_{k_p}(x) = u^p + k_p(u)(x_1 - u).$$

A_{k_p} is even w.r.t. x_1:

$$A_{k_p}(-x_1, x_2) = A_{k_p}(x_1, x_2), \quad x \in \omega.$$

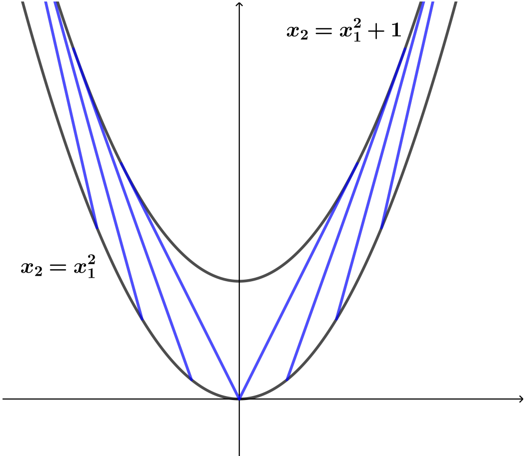
In the "cup", $x_1^2 \leq x_2 \leq 1$:

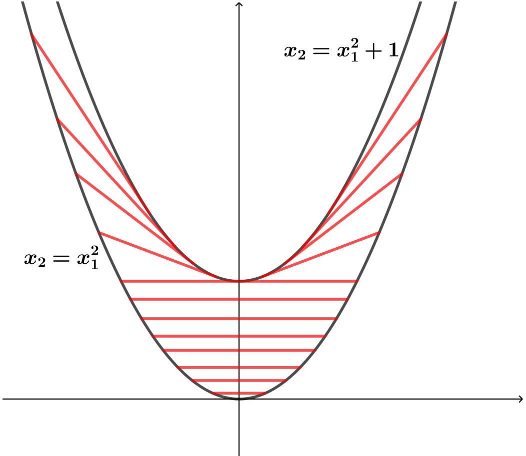
$$A_{k_p}(x_1, x_2) = x_2^{p/2}.$$



Description of $B_p^{\pm}$ by Slavin and Vasyunin

$$\boldsymbol{B}_p^{+} = A_{m_p}, \qquad p \in [2, +\infty), \qquad \boldsymbol{B}_p^{-} = A_{k_p};$$
$$\boldsymbol{B}_p^{-} = A_{m_p}, \qquad p \in [1, 2], \qquad \boldsymbol{B}_p^{+} = A_{k_p}.$$





Properties of $B_{p,r}$

- $\boldsymbol{B}_{p,r}$ is defined on

$$\Omega = \Big\{x \in \mathbb{R}^3 \colon x_1^2 \le x_2 \le x_1^2 + 1,\ \boldsymbol{B}_p^-(x_1, x_2) \le x_3 \le \boldsymbol{B}_p^+(x_1, x_2)\Big\};$$

- Boundary condition:

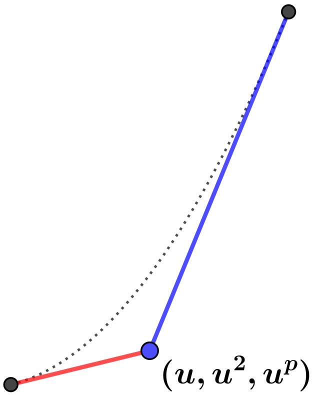
$$\boldsymbol{B}_{p,r}(t, t^2, |t|^p) = |t|^r; \tag{8}$$

- $\boldsymbol{B}_{p,r}$ is locally concave on Ω;
- $\boldsymbol{B}_{p,r}$ is the (pointwise) minimal locally concave on Ω satisfying (8);
- $\boldsymbol{B}_{p,r}$ is linear along the foliation on the boundaries;
- For $(x_1, x_2) \in \omega$ one has

$$\boldsymbol{B}_{p,r}(x_1, x_2, A_{m_p}(x_1, x_2)) = A_{m_r}(x_1, x_2),$$
$$\boldsymbol{B}_{p,r}(x_1, x_2, A_{k_p}(x_1, x_2)) = A_{k_r}(x_1, x_2).$$

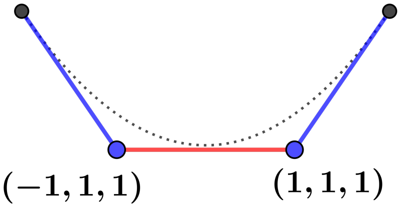
Construction. Simple case

For $u > 1$ draw red and blue segments in $\mathbb{R}^3$ starting at (u, u^2, u^p) and take G to be linear on a curvilinear triangle. The same for $u < -1$:



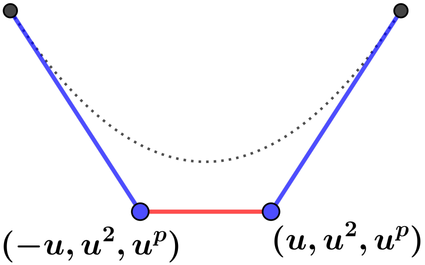
Construction. Simple case

For $u = 1$ two such triangles meet and form a curvilinear quadrilateral:



Construction. Simple case

For $u \in (0,1]$ take G to be linear on a curvilinear quadrilateral:



Construction. Simple case

Direct calculations show that if $(r-p)(r-2) \geq 0$, then this function G is locally concave. Thus, we have $\boldsymbol{B}_{p,r} \leq G$.

On the other hand, on each "blue-red" figure we have $\boldsymbol{B}_{p,r} \geq G$. Therefore, $\boldsymbol{B}_{p,r} = G$.

The domain Ω is foliated by 2-dimensional "blue-red" figures, on each of them $\boldsymbol{B}_{p,r}$ is linear.

For the case $(r-p)(r-2) < 0$ G is locally convex. Construction of the minimal locally concave function is much harder, the foliation consists mostly of 1-dimensional figures of linearity (segments).

The function $\boldsymbol{B}_{p,r}$ is constructed, Theorem 1 is proved.

Transference to $\mathbb{T}$ and $\mathbb{R}$

Transference to $\mathbb{T}$

$$\langle|\varphi|^r\rangle_\Omega \leq C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|^{r-p}_{\mathrm{BMO}(\Omega)}, \qquad \langle\varphi\rangle_\Omega = 0. \tag{2}$$

Inequality (2) for $\Omega = \mathbb{T}$ trivially follows from (2) for $\Omega = [0,1]$ with the same constant. To prove its sharpness we use a special procedure to transfer the optimizer from $\mathrm{BMO}([0,1])$ to $\mathbb{T}$ almost preserving its distribution (following Stolyarov-Z. based on F. Nazarov's idea).

Transference to $\mathbb{R}$

$$\langle|\varphi|^r\rangle_\Omega \leq C(p,r;\Omega)\cdot\langle|\varphi|^p\rangle_\Omega\cdot\|\varphi\|^{r-p}_{\mathrm{BMO}(\Omega)}. \tag{3}$$

Inequality (3) for $\Omega = \mathbb{R}$ follows from (2) for $\Omega = [0,1]$ by a simple limiting argument: apply (2) for $\varphi - \langle\varphi\rangle_{[-n,n]}$ and tend n to $+\infty$. To prove sharpness we take the optimizer on $\mathbb{T}$, consider it as a 1-periodic function on $\mathbb{R}$, and then modify it.

Dimension-free estimate via heat norm (following Slavin–Z.)

Let W_s be a Wiener process in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $t > 0$ let $Z_s = (x + W_s, t - s), s \in [0, t]$. For any $\psi\colon \mathbb{R}^d \to \mathbb{R}$ remind the definition:

$$\psi_H(x, t) = (H_t * \psi)(x).$$

$\psi_H(Z_s)$ is a martingale. Apply for $\psi = \varphi, \varphi^2, |\varphi|^p$, for a fixed φ with $\|\varphi\|_H \leq 1$. $\boldsymbol{B}_{p,r}$ is locally concave, therefore,

$$\mathbb{E}\boldsymbol{B}_{p,r}(\varphi_H(Z_s), \varphi_H^2(Z_s), |\varphi|_H^p(Z_s))$$

decreases in s. It follows that

$$\boldsymbol{B}_{p,r}(\varphi_H(x,t), \varphi_H^2(x,t), |\varphi|_H^p(x,t)) \geq \mathbb{E}\boldsymbol{B}_{p,r}(\varphi_H(Z_t), \varphi_H^2(Z_t), |\varphi|_H^p(Z_t)) = \mathbb{E}|\varphi_H|^r(Z_t) = |\varphi_H|^r(x,t).$$

Apply for $\varphi - \varphi_H(x,t)$:

$$C(p, r, [0,1]) \cdot |\varphi - \varphi_H(x,t)|_H^p(x,t) \geq |\varphi - \varphi_H(x,t)|_H^r(x,t).$$

Multiplying by $t^{d/2}$ and tending $t \to +\infty$, we obtain

$$C(p, r, [0,1]) \cdot \|\varphi\|_{L^p}^p \geq \|\varphi\|_{L^r}^r.$$

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Thanks

Thanky you for your attention!