

Realisation of groups as automorphism groups of maps and hypermaps

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Realisation Problem: Given an abstract group A and a class \mathcal{C} of mathematical objects, is $A \cong \text{Aut}_{\mathcal{C}} \mathcal{O}$ for some object $\mathcal{O} \in \mathcal{C}$?

Theorem (Frucht, 1939)

Every finite group is isomorphic to the automorphism group of a finite graph.

Theorem (Sabidussi, 1960)

Every group is isomorphic to the automorphism group of a graph.

There are similar results for many other classes of objects, e.g. Riemann surfaces, fields, hyperbolic manifolds, polytopes, etc, with only the obvious cardinality constraints. For example:

Theorem (Cori and Machì, 1982)

Every finite group is isomorphic to the automorphism group of a finite oriented hypermap (= dessin d'enfant, Grothendieck, 1984).

What about unoriented hypermaps, or those of a given type?

Theorem (J., *Ars Math. Contemp.*, 20 (2021), online)

For $p, q, r \in \mathbb{N} \cup \{\infty\}$, $p^{-1} + q^{-1} + r^{-1} < 1$, let \mathfrak{C} be the category of all *hypermaps*, or *oriented hypermaps*, of type dividing (p, q, r) .

a) Every *finite group* A is isomorphic to $\text{Aut}_{\mathfrak{C}} \mathcal{O}$ for \aleph_0 non-isomorphic *finite* objects $\mathcal{O} \in \mathfrak{C}$.

b) If p, q or $r = \infty$, every *countable group* A is isomorphic to $\text{Aut}_{\mathfrak{C}} \mathcal{O}$ for 2^{\aleph_0} non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$.

By taking $q = 2$ we get the corresponding results for *maps*.

The hyperbolic condition $p^{-1} + q^{-1} + r^{-1} < 1$ cannot be omitted: these results fail badly for spherical and euclidean types (p, q, r) .

Conjecture

The condition “If p, q or $r = \infty$ ” can be removed from (b).

Method of proof

Let Γ be the extended or ordinary triangle group $\Delta[p, q, r]$ or $\Delta(p, q, r)$ corresponding to the category \mathfrak{C} .

Connected objects $\mathcal{O} \in \mathfrak{C}$ correspond to conjugacy classes of subgroups $M \leq \Gamma$, with $\text{Aut}_{\mathfrak{C}} \mathcal{O} \cong N_{\Gamma}(M)/M$.

If a group A has rank $d \in \mathbb{N} \cup \{\aleph_0\}$, find a **maximal subgroup** N of Γ with **epimorphisms** $N \rightarrow F_d$ (free group of rank d), and $F_d \rightarrow A$ so that $M := \ker(N \rightarrow F_d \rightarrow A)$ is not normal in Γ . Then $N_{\Gamma}(M) = N$ by the maximality of N , and $N/M \cong A$.

In general, choosing $F_d \rightarrow A$ is easy but finding N is harder, especially when $d = \aleph_0$ so that N must have infinite index in Γ .

However, if N exists there are usually 2^{\aleph_0} (or \aleph_0 if A is finite) possibilities for M , mutually non-conjugate in Γ , giving that many non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$ realising A as $\text{Aut}_{\mathfrak{C}} \mathcal{O}$.

Finding maximal subgroups: Case (a), A finite.

Let \mathfrak{C} be the **oriented** category, so $\Gamma = \Delta(p, q, r)$. If $p, q, r < \infty$, map Γ smoothly onto $\mathrm{PSL}_2(\mathbb{F}_n)$ for (infinitely many) primes $n \equiv -1 \pmod{\ell}$, where $\ell := \mathrm{lcm}(2p, 2q, 2r)$, giving \aleph_0 2-transitive (hence primitive) representations of Γ of degrees $n + 1$.

Point-stabilisers $N \leq \Gamma$ are maximal, of finite index, and are surface groups of genus $g \rightarrow \infty$ as $n \rightarrow \infty$, so they map onto F_g , and F_g maps onto A for large enough n .

The subgroups $M = \ker(N \rightarrow F_g \rightarrow A)$ have index $(n + 1)|A| < \infty$ in Γ , so they correspond to finite objects $\mathcal{O} \in \mathfrak{C}$ with $\mathrm{Aut}_{\mathfrak{C}} \mathcal{O} \cong A$.

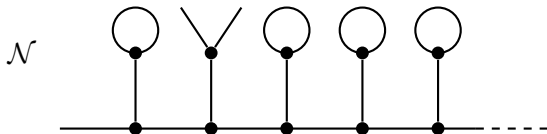
If p, q or $r = \infty$, first map Γ onto $\Delta(p', q', r')$ with finite periods and then do as above. (Parabolic generators of N are no problem.)

If \mathfrak{C} is the **unoriented** category, so $\Gamma = \Delta[p, q, r]$, the proof is similar, but using NEC groups instead of Fuchsian groups.

Finding maximal subgroups: Case (b), $|A| \leq \aleph_0$, $r = \infty$.

Now we need epimorphisms $N \rightarrow F_\infty$, so N must not be finitely generated. Thus $|\Gamma : N| = \infty$, so we need primitive representations of **infinite** degree. If (wlog) $r = \infty$ then $\Gamma = \Delta(p, q, r) \cong C_p * C_q$, and we can generalise results of B. H. Neumann, Magnus, Lyndon, etc for the modular group $C_3 * C_2$ (J., *J. Group Theory*, 2019).

Example $\mathfrak{C} = \{\text{oriented cubic maps}\}$, $\Gamma = \Delta(3, 2, \infty) \cong \text{PSL}_2(\mathbb{Z})$.



One can show that the monodromy group of \mathcal{N} is primitive on arcs, so the map subgroup $N = \Gamma_\alpha \cong C_2 * C_2 * C_2 * F_\infty$ is maximal in Γ .

For any countable $A \neq 1$ there are epimorphisms $N \rightarrow F_\infty \rightarrow A$ with 2^{\aleph_0} non-conjugate kernels M , not normal in Γ . These correspond to 2^{\aleph_0} non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$ with $\text{Aut}_{\mathfrak{C}} \mathcal{O} \cong A$.

Similar arguments, with more complicated \mathcal{N} , apply to $A = 1$ and to any $\Delta(p, q, \infty) \cong C_p * C_q$ or $\Delta[p, q, \infty]$ with $p^{-1} + q^{-1} < 1$.

What if $|A| = \aleph_0$ and p, q, r are all finite?

Theorem (J, AMC 20 (2021))

b') *If $p, q, r \in \mathbb{N}$ with $3 \mid p, 2 \mid q$ and $r \geq 7$, every countable group A is isomorphic to $\text{Aut}_{\mathfrak{C}} \mathcal{O}$ for 2^{\aleph_0} non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$.*

(A proportion 121/216 of all finite triples satisfy this condition.)

Outline proof Marston Conder (1981) constructed orientable finite hypermaps of type $(3, 2, r)$, $r \geq 7$, with monodromy group A_n (n large) by sewing finitely many coset diagrams together. Sewing **infinitely many** gives 2^{\aleph_0} infinite hypermaps of this type and of infinite genus with infinite primitive monodromy groups, and hence 2^{\aleph_0} maximal subgroups of infinite index and genus in $\Delta(3, 2, r)$. Lift these back to subgroups N of $\Delta(p, q, r)$ with the same properties, and argue as before to realise all countable groups A . \square

Theorem (J, AMC 20 (2021))

b') *If $p, q, r \in \mathbb{N}$ with $3 \mid p$, $2 \mid q$ and $r \geq 7$, every countable group A is isomorphic to $\text{Aut}_{\mathfrak{C}} \mathcal{O}$ for 2^{\aleph_0} non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$.*

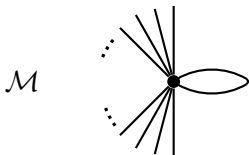
Conjectured extension to all hyperbolic types (p, q, r) :

Conjecture

b'') *If $p, q, r \in \mathbb{N}$ with $p^{-1} + q^{-1} + r^{-1} < 1$, every countable group A is isomorphic to $\text{Aut}_{\mathfrak{C}} \mathcal{O}$ for 2^{\aleph_0} non-isomorphic objects $\mathcal{O} \in \mathfrak{C}$.*

Evidence Brent Everitt (2000) et al. extended Marston Conder's construction of finite hypermaps with monodromy groups A_n to **all hyperbolic** types (p, q, r) ; a similar extension to infinite hypermaps, giving primitive monodromy groups of infinite degree, works in some (and possibly all) cases, though the details are much more complicated than for type $(3, 2, r)$.

Simultaneous realisation



Theorem (J, 2021)

Each finite group A is embedded in the monodromy group G of the oriented map \mathcal{M} ($\cong \text{Aut}^+ \tilde{\mathcal{M}}$ where $\tilde{\mathcal{M}}$ is its regular cover).

Proof Labelling the arcs of \mathcal{M} with the integers, we have

$$G = \langle x : t \mapsto t + 1, y = (0, 1) \rangle \leq \text{Sym } \mathbb{Z},$$

so $G = F \rtimes C$ where F (the normal closure of y) is the group of all finitary permutations of \mathbb{Z} , and $C = \langle x \rangle \cong C_\infty$. By Cayley's Theorem, if $|A| = n < \infty$ then $A \leq S_n \leq F \leq G$. □

For background, see:

J, Realisation of groups as automorphism groups in categories,
arXiv:1807.00547 [math.GR], *AMC* 20 (2021), online.

J, Maximal subgroups of the modular and other groups,
arXiv:1806.03871 [math.GR], *J. Group Theory* 22 (2019), 277–296.

The first of these has some small overlap with:

R. Bottinelli, L. Grave de Peralta and A. Kolpakov,
Telescopic groups and symmetries of combinatorial maps,
arXiv:1901.05710 [math.CO], *Algebr. Comb.* 3 (2020), 483–508.

THANK YOU FOR LISTENING!