Almost mixed Moore graphs and their spectra

8ECM

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Degree/Diameter problem for undirected graphs

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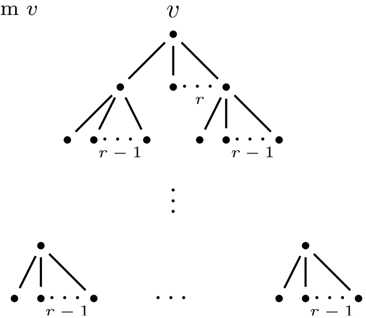

distance from v	number of vertices
0	1
1	r
2	$r(r-1)$
$\vdots$	$\vdots$
$k-1$	$r(r-1)^{k-2}$
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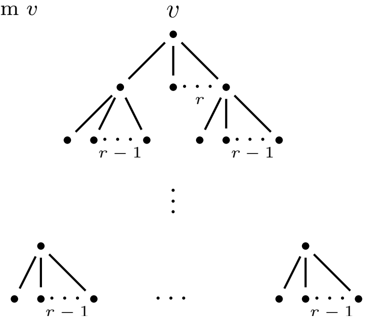

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(Moore bound for undirected graphs)

$$n(r,k) \leq M(r,k) = 1 + r + r(r-1) + \cdots + r(r-1)^{k-1}.$$

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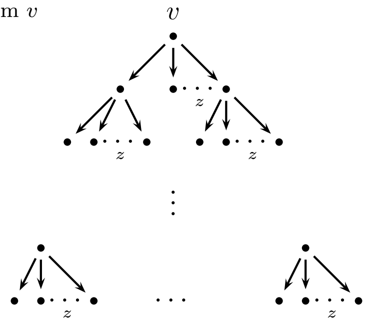

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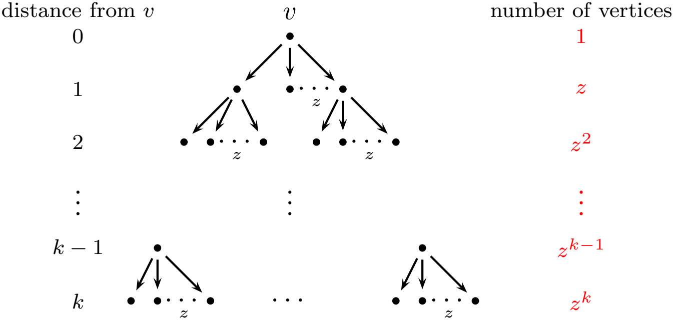

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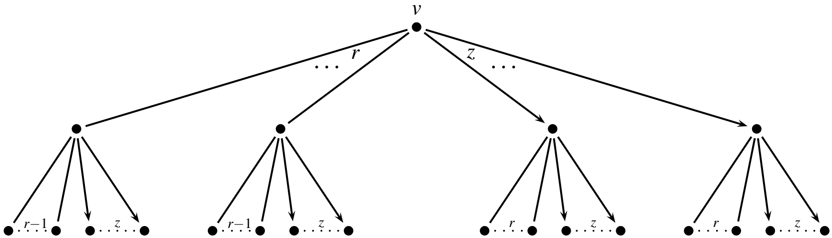

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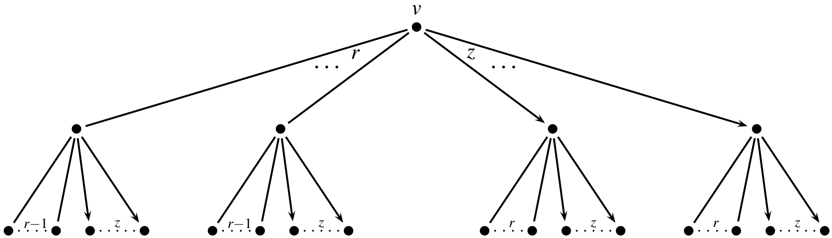

Diameter $k = 2$:
$n(r,z,2) \leq M(r,z,2) = 1 + (r+z) + z(r+z) + r(r+z-1).$

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- $z = 0$ (no arcs) $\rightarrow$ *Moore graphs* [Banai and Ito '73, Hoffman and Singleton '60, Damerell '73] only exists for $k = 1$ and $r \geq 1$ (Complete graph K_{r+1}) or $k \geq 3$ and $r = 2$ (Cycle graph C_{2k+1}) or
  - $k = 2$ and $r = 2$ (Cycle graph C_5);
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- $r = 0$ (no edges) → *Directed Moore graphs* [Plesník and Znám '74, Bridges and Toueg '80] only exists for $k = 1$ (complete digraph K_{z+1}) or $z = 1$ (directed cycle $\overrightarrow{C}_{k+1}$).

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Let G be a (proper) mixed Moore graph of diameter 2*. Then, G is totally regular with directed degree* $z \geq 1$ *and undirected degree* $r \geq 1$*. Moreover, there must exist a positive odd integer c such that*

$$\textit{Exists odd } c \in \mathbb{Z} \textit{ such that } r = \frac{1}{4}(c^2+3) \textit{ and } c \mid (4z-3)(4z+5)$$

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$$I + A + A^2 = J + rI$$

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Let $d = r + z$ (total degree). With the mapping $x \to 1 + x + x^2$ and the Moore bound $n = 1 + d + d^2 - r$ we obtain that the factors of $\Phi_A(x)$ are $(x-d)$ and the factors of $(1+x+x^2-r)^{\frac{n-1}{2}}$. Two cases to analyze, depending on the irreductibility of $P(x) = (1+x+x^2-r)$ in $\mathbb{Q}$:

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- $P(x) = (1 + x + x^2 - r)$ is reducible in $\mathbb{Q}$: Then $4r - 3$ must be a square c^2 in $\mathbb{Z}$. Therefore, the roots of $P(x)$ are $\alpha = \frac{-1+c}{2}$ and $\beta = \frac{-1-c}{2}$. As a consequence,

$$\Phi_A(x) = (x-d)(x-\alpha)^a(x-\beta)^b,$$

where a and b are completely determined by the degrees r and z. Equation $\mathrm{Tr}(A) = 0$ gives the necessary condition given above. ☐

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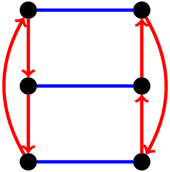

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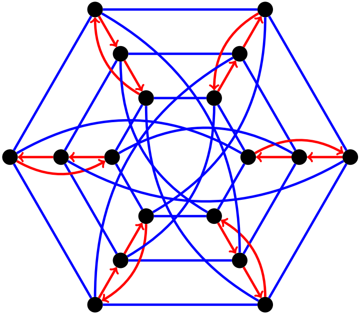

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3. Finite geometries: Biaffine planes

Bosák graph constructed by voltage assignament

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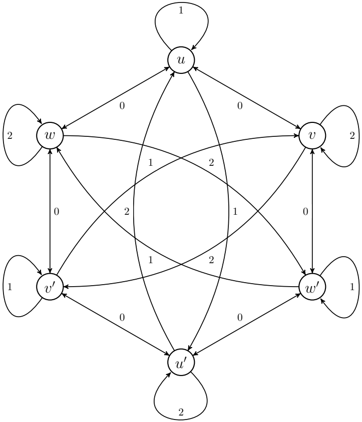

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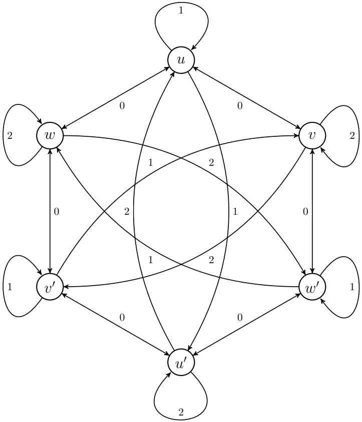

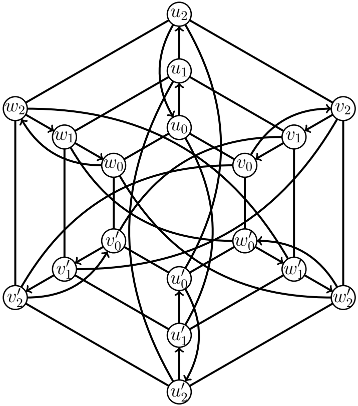

$G \quad \rightarrow \quad G^{\alpha}$

Mixed Moore graphs of diameter 2. Feasible cases. 1978

[Bosák] Exists odd $c \in \mathbb{Z}$ such that $r = \frac{1}{4}(c^2 + 3)$ and $c | (4z - 3)(4z + 5)$

$M(r,z,2)$	r	z	d	Existence	Uniqueness
6	1	1	2	Yes	?
12	1	2	3	Yes	?
18	3	1	4	Yes	?
20	1	3	4	Yes	?
30	1	4	5	Yes	?
40	3	3	6	?	?
42	1	5	6	Yes	?
54	3	4	7	?	?
56	1	6	7	Yes	?
72	1	7	8	Yes	?
84	7	2	9	?	?
88	3	6	9	?	?
90	1	8	9	Yes	?
108	3	7	10	?	?
110	1	9	10	Yes	?
⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2001

[Gimbert] Enumeration of almost Moore digraphs of diameter 2

$M(r,z,2)$	r	z	d	Existence	Uniqueness
6	1	1	2	Ka$(2,2)$	Yes
12	1	2	3	Ka$(3,2)$	Yes
18	3	1	4	Yes	?
20	1	3	4	Ka$(4,2)$	Yes
30	1	4	5	Ka$(5,2)$	Yes
40	3	3	6	?	?
42	1	5	6	Ka$(6,2)$	Yes
54	3	4	7	?	?
56	1	6	7	Ka$(7,2)$	Yes
72	1	7	8	Ka$(8,2)$	Yes
84	7	2	9	?	?
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⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2007

[Nguyen, Miller, Gimbert] On mixed Moore graphs

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6	1	1	2	$\mathrm{Ka}(2,2)$	Yes
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⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2014

[L., Pérez-Rosés, Pujolàs] mixed Moore Cayley graphs

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Mixed Moore graphs of diameter 2. Feasible cases. 2015

[Jørgensen] New mixed Moore graphs and directed strongly regular graphs

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⋮	⋮	⋮	⋮	⋮	⋮

Mixed Moore graphs of diameter 2. Feasible cases. 2016

[L., Miret, Fern.] Non existence of some mixed Moore graphs using SAT

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Mixed Moore graphs of diameter 2. Feasible cases.

Exists odd $c \in \mathbb{Z}$ such that $r = \frac{1}{4}(c^2+3)$ and $c|(4z-3)(4z+5)$

$M(r,z,2)$	r	z	d	Existence	Uniqueness
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108	3	7	10	Jørgensen graphs	No
110	1	9	10	$\mathrm{Ka}(10,2)$	Yes
132	1	10	11	$\mathrm{Ka}(11,2)$	Yes
150	7	5	12	?	?
154	3	9	12	?	?
156	1	11	12	$\mathrm{Ka}(12,2)$	Yes
180	3	10	13	?	?
182	1	12	13	$\mathrm{Ka}(13,2)$	Yes
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Problem

Prove or dismiss the existence of mixed Moore graphs.

Mixed Moore graphs of diameter $k \geq 3$.

Question (Bosak, 1978)

Are there mixed Moore graphs of diameter $k \geq 3$*?*

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The regularity question has been studied recently by J. Tuite and G. Erskine, showing that every mixed graph of

- diameter $k = 2$ and order $M(r, z, 2) - 1$;
- degrees $r = z = 1$ and order $M(1, 1, k) - 1$

must be regular.

Mixed almost Moore graphs

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Every mixed almost Moore graph G of diameter two has the property that for each vertex $v \in V(G)$ there exists only one vertex, denoted by $\sigma(v)$ and called the **repeat** of v, such that there are exactly two walks of length at most 2 from v to $\sigma(v)$.

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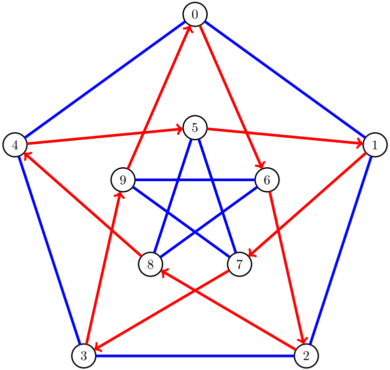
Proposition

G is a totally regular graph if and only if P is a permutation matrix (the map σ is an automorphism of G).

Seeing σ as a permutation, it has a *cycle structure* which corresponds to its unique decomposition in disjoint cycles. Such cycles will be called *permutation cycles* of G. The number of permutation cycles of G of each length $i \leq n$ will be denoted by m_i and the vector $(m_1, \ldots, m_n)$ will be referred to as the *permutation cycle structure* of G.

Mixed almost Moore graphs of diameter 2. Example



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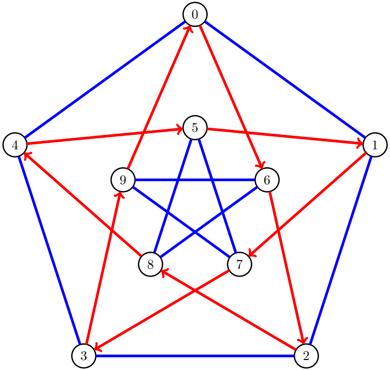

Automorphism of repeats: $\sigma = (0\ 2\ 4\ 1\ 3)\,(5\ 7\ 9\ 6\ 8)$.

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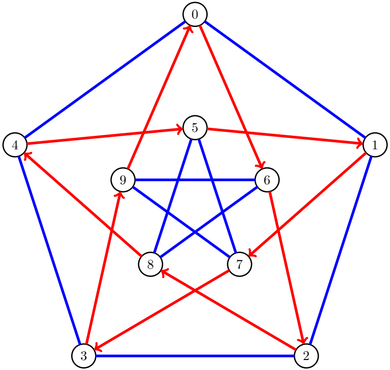

Automorphism of repeats: $\sigma = (0\ 2\ 4\ 1\ 3)\ (5\ 7\ 9\ 6\ 8)$.

Permutation cycle structure: $(m_1, \ldots, m_{10})$ is $m_5 = 2$ and $m_i = 0$ $(i \neq 5)$.

Spectra of $J+P$ and the cyclotomic polynomials

Let P a permutation matrix with permutation cycle structure $(m_1,\ldots,m_n)$. Then,

$$\det\big(xI-(J+P)\big)=\big(x-(n+1)\big)(x-1)^{\sum_{i=1}^{n}m_i-1}\prod_{i=2}^{n}(x^i-1)^{m_i}.$$

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Since $x^l-1=\prod_{i|l}\Phi_i(x)$, where $\Phi_i(x)$ denotes the i-th cyclotomic polynomial, the factorization of $\det\big(xI-(J+P)\big)$ in $\mathbb{Q}$ is

$$\det\big(xI-(J+P)\big)=\big(x-(n+1)\big)(x-1)^{m(1)-1}\prod_{i=2}^{n}\Phi_i(x)^{m(i)},$$

where $m(i)=\sum_{i|l}m_l$ represents the total number of permutation cycles of order multiple of i.

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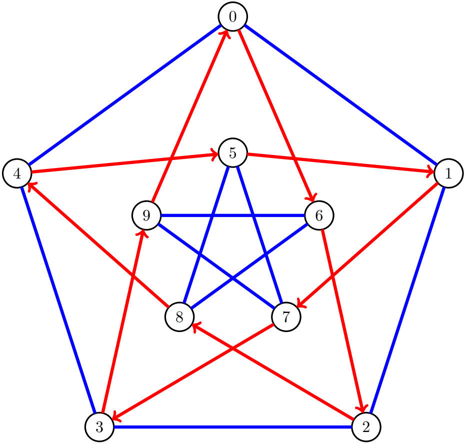
$$\det\big((x-r)I-(J+P)\big) = \big(x-(n+r+1)\big)\big(x-(r+1)\big)^{m(1)-1}\prod_{i=2}^{n}\Phi_i(x-r)^{m(i)}.$$

We just need to add mapping $x \to 1+x+x^2$ to have information about the factors of the characteristic polynomial of G.

Proposition

The irreducible factors of the characteristic polynomial $\phi_G(x)$ of a mixed almost Moore graph G are $(x-d)$ and some of the irreducible factors of $\Phi_i\big(x^2+x-(r-1)\big)$, for all $1 \leq i \leq n$.

Example



$$\phi_G(x) = (x-3)(x-1)\Phi_5(x^2+x-1)$$

Irreducibility of $\Phi_i\big(x^2+x-(r-1)\big)$

1. $\Phi_1\big(x^2+x-(r-1)\big)=x^2+x-r$ is irreducible in $\mathbb{Q}$ iff $4r+1$ is not a square in $\mathbb{Z}$.

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Theorem

The polynomials $\Phi_i\big(x^2+x-(r-1)\big)$ *are irreducible in* $\mathbb{Q}$ *for all* $i\geq 3$ *and* $r\geq 1$.

Mixed almost Moore graphs

Theorem (L. and Miret, '15)

Let G be a (totally regular) mixed almost Moore graph of diameter 2, *undirected (even) degree* $r > 2$, *and directed degree* $z \geq 1$. *Then,*

- *There exists* $c_1 \in \mathbb{Z}$ *such that* $c_1^2 = 4r + 1$ *and* $c_1 \mid (4z+1)(4z-7)$.
- *There exists* $c_2 \in \mathbb{Z}$ *such that* $c_2^2 = 4r - 7$ *and* $c_2 \mid (16z^2 + 40z - 23)$.

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r	c_1	c_2	z	n	Existence
4	-	3	$1, 4, 7, 10, \ldots$	$26, 68, 128, 206, \ldots$	Unknown
6	5	-	$1, 3, 6, 8, \ldots$	$50, 84, 150, 204, \ldots$	Unknown
8	-	5	-	-	non-existent
10	-	-	-	-	non-existent
12	7	-	$5, 7, 12, 14, \ldots$	$294, 368, 588, 690, \ldots$	Unknown
14	-	7	-	-	non-existent
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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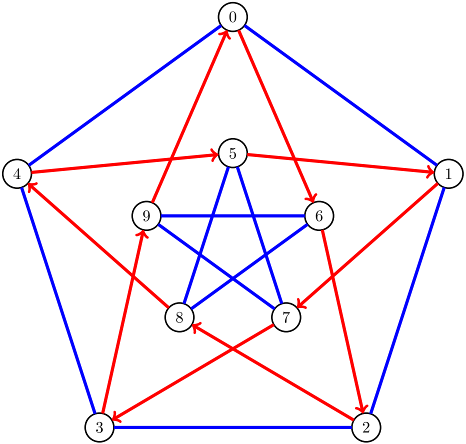
5. Compute the trace of the powers of A in two ways: geometrically and using $\phi_G(x)$.
6. Compare both results and derive conditions for the existence of G.

Mixed almost Moore graphs

- The unique mixed almost Moore graph for $r = 2$ and $z = 1$



$$\phi_G(x) = (x-3)(x-1)\Phi_5(x^2+x-1)$$