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Level densities for general β-ensembles: An operator-valued free probability perspective

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Outline of the presentation

- Setup for the talk
- General β-ensembles and level densities
- Generalized multiv. Fuss-Narayana polynomials and general level correlation function
- Level densities with FN-type polynomials
- Application: Sampling general β-ensembles
- Extensions and conclusion

Setup for the talk – Invariant ensembles: unitary, orthogonal, symplectic

- Let $\mathrm{Her}(N)$ denote the space of $N \times N$ Hermitian matrices and $U(N)$ the group of $N \times N$ unitary matrices. A probability distribution $\mathbb{P}$ on $\mathrm{Her}(N)$ is **invariant** if for every fixed $V \in U(N)$ and every Borel set A in $\mathrm{Her}(N)$ we have $\mathbb{P}(M \in A) = \mathbb{P}(VMV^* \in A)$.
- Invariant ensembles of matrices have probability distributions of form:

$$\mathcal{P}_{N,\beta}(M)dM = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-trV_\beta(M)} dM$$

  for $\beta = 1,2,4$ – the so-called **Orthogonal ($\boldsymbol{\beta = 1}$)**, **Unitary ($\boldsymbol{\beta = 2}$)** and **Symplectic ($\boldsymbol{\beta = 4}$)** ensembles. For $\beta = 1,2,4$ the ensembles consist of, respectively, $N \times N$ real symmetric matrices, $N \times N$ Hermitian matrices and $2N \times 2N$ Hermitian self-dual matrices.
- In general the potential $V_\beta(x)$ is a real-valued function growing sufficiently rapidly as $|x| \to \infty$, but largely we study V_β's which are polynomials, $V_\beta(x) = \kappa_{2m,\beta} x^{2m} + \cdots$, $\kappa_{2m,\beta} > 0$.

Setup for the talk: determinantal processes

- For **UIE**, the **eigenvalue distribution** is:

$$p(\lambda_1,\dots,\lambda_N)d\lambda_1 \dots d\lambda_N = \frac{1}{Z_N}\Delta_N(\lambda)^2\prod_{j=1}^{N} e^{-V(\lambda_j)}\, d\lambda_j = \frac{1}{Z_N}\prod_{i<j}\left(\lambda_i-\lambda_j\right)^2 w(\lambda_1)\dots w(\lambda_n)d\lambda$$

$$= \frac{1}{N!}det\begin{pmatrix}\mathcal{K}_N(\lambda_1,\lambda_1) & \cdots & \mathcal{K}_N(\lambda_N,\lambda_1)\\ \vdots & \ddots & \vdots \\ \mathcal{K}_N(\lambda_1,\lambda_N) & \cdots & \mathcal{K}_N(\lambda_N,\lambda_N)\end{pmatrix}d\lambda$$

$$\mathcal{K}_N(x,y) = \sum_{k=0}^{N-1}\phi_k(x)\phi_k(y) \qquad\qquad Z_{N,\beta} = (2\pi)^{N/2}\prod_{j=1}^{N}\frac{\Gamma(1+\frac{j\beta}{2})}{\Gamma(1+\frac{\beta}{2})}$$

- This is a **determinantal point process** with correlation kernel $K_N{}^{(2)}(x,y) = \sum_{k=0}^{N-1} p_k(x)\, p_k(y) e^{-\frac{1}{2}(V(x)+V(y))}$
- Using **basic Christoffel-Darboux** for UIE $K_N{}^{(2)}(x,y) = \frac{\Phi^{(2)t}(x)[Q^{(2)},\Pi_N]\Phi^{(2)}(y)}{(x-y)}$

$$K_N{}^{(2)}(x,y) = \frac{u_{N-1}}{u_N}\frac{p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)}{x-y}$$

- The **correlation kernel for UIE** can also be written as: $K_N{}^{(2)}(x,y) = \sum_{j=0}^{N-1}\phi_j{}^{(2)}(x)\,\phi_j{}^{(2)}(y)$, $\phi_j{}^{(2)}(x) = \sqrt{\frac{w(x)}{h_j}}P_j(x)$ where the $P_j(x)$ are **orthogonal polynomials** of order j defined by $\int_{\mathbb{R}} P_n(x)\, P_m(x) w(x) dx = h_n\delta_{n,m}$, $n,m \in \mathbb{N}$, $w(x)$ is the weight function with respect to which they are defined, and h_j the normalization constant.

About Pfaffian processes and skew orthogonal polynomials

- The eigenvalue point process for GOE and GSE is not determinantal but **Pfaffian point process**:

$$\mathrm{Pf}(A) = \sum_{\substack{\sigma\in S_{2N}\\ \sigma_{2i-1}<\sigma_{2i}}} (-1)^{|\sigma|} \prod_{i=1}^{N} A_{\sigma_{2i-1},\sigma_{2i}} \qquad p(x_1,\dots,x_N) = \frac{1}{Z_{2N}} \det[f_j(x_k)]_{j,i=1}^{2N} \mathrm{Pf}[\varepsilon(x_j,x_k)]_{j,k=1}^{2N}$$

$$\rho^{(n)}(x_1,\dots,x_N) = \mathrm{Pf}[\widetilde{K}(x_i,x_j)]_{i,j=1}^{n} \qquad \widetilde{K}(x,y) = \begin{pmatrix} \widetilde{K}_1(x,y) & \widetilde{K}_2(x,y) \\ \widetilde{K}_3(x,y) & \widetilde{K}_4(x,y) \end{pmatrix}$$

- **Generalized Christoffel-Darboux:**

$$\textbf{OIE: } {K_N}^{(1)}(x,y) = \frac{\widehat{\Psi}^{(1)}(x)[\bar{R}^{(1)}(y),\Pi_{2N}]\Psi^{(1)}(y)}{f(x)(x-y)}, \qquad N \geq d$$

$$\textbf{SIE: } {K_N}^{(4)}(x,y) = \frac{\widehat{\Phi}^{(4)}(x)[\bar{R}^{(4)}(x),\Pi_{2N}]\Phi^{(4)}(y)}{f(y)(x-y)}, \qquad N \geq d$$

- **Skew-orthogonal polynomials for OIE and SIE:**

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \pi_n^{(1)}(x)\,\pi_m^{(1)}(y)\epsilon(x-y)w(y)w(x)dydx = g_n^{(1)} Z_{n,m}, \qquad \epsilon(r) = \frac{|r|}{2r}$$

$$\int_{\mathbb{R}} [\pi_n^{(4)}(x){\pi_m^{(4)}}'(x) - \pi_m^{(4)}(x){\pi_n^{(4)}}'(x)]w^2(x)dx = g_n^{(4)} Z_{n,m}, \qquad (') = \frac{d}{dx}$$

$$f(x)\Psi_n^{(4)}(x) = \mathbf{P}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{P}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{P}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x)$$

$$xf(x)\Psi_n^{(4)}(x) = \mathbf{R}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{R}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{R}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x)$$

$$f(x)\Phi_n^{(1)}(x) = \mathbf{P}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{P}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{P}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x)$$

$$xf(x)\Phi_n^{(1)}(x) = \mathbf{R}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{R}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{R}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x)$$

- **Universality properties for OIE and SIE** have been proven for certain cases by Delft and Gioev (2004 – in the bulk); Delft and Gioev (2005 – at the edge); and separately by Ghosh (2009 – in the bulk, using the generalized Christoffel –Darboux formula and the asymptotic results of skew-orthogonal functions). The general case has been resolved in Kriecherbauer and Shcherbina (2010) and Shcherbina (2011).

General β-ensembles

- I. Dumitriu and A. Edelman (2002): Classical Random Matrix Theory focuses on the random matrix models in the following 3×3 table:

Random Matrix Ensembles

Joint Eigenvalue Distributions: $c\,|\Delta|^{\beta} \prod_{i=1}^{n} V(\lambda_i)$

Technical name	Traditional name	β	Field	Property	Invariance	Connected Matrix Problem
Hermite ensembles $V(\lambda) = e^{-\lambda^2/2}$	**Gaussian ensembles** GOE GUE GSE	 1 2 4	 $\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$	 Symmetric Hermitian Self–Dual	 $A \to Q^T A Q$ $A \to U^H A U$ $A \to S^D A S$	(EIG) Symmetric Eigenvalue Problem
Laguerre ensembles $V(\lambda) = \lambda^a e^{-\lambda/2}$ $a = \frac{\beta}{2}(n - m + 1) - 1$	**Wishart ensembles** Wishart real Wishart complex Wishart quaternion	 1 2 4	 $\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$	Positive Semi–Definite	 $A \to Q^T A Q$ $A \to U^H A U$ $A \to S^D A S$	(SVD) Singular Value Decomposition
Jacobi ensembles $V(\lambda) = \lambda^a (1-\lambda)^b$ $a = \frac{\beta}{2}(n_1 - m + 1) - 1$ $b = \frac{\beta}{2}(n_2 - m + 1) - 1$	**MANOVA ensembles** MANOVA real MANOVA complex MANOVA quaternion	 1 2 4	 $\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$	Positive Semi–Definite	$X \to Q^T X Q_1$ $Y \to Q^T Y Q_1$ $X \to U^H X U_1$ $Y \to U^H Y U_1$ $X \to S^D X S_1$ $Y \to S^D Y S_1$	(QZ) Generalized Symmetric Eigenvalue Problem

General β-ensembles

- **The three β-Gaussian/Hermite Ensembles have joint eigenvalue probability density function:**

$$f_\beta(\lambda) = c_H^\beta \prod_{i<j} \left|\lambda_i - \lambda_j\right|^\beta e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \text{ with } c_H^\beta = (2\pi)^{-n/2} \prod_{j=1}^n \frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}j\right)}$$

- **β-Laguerre/Wishart case:**

$$f_\beta(\lambda) = c_L^{\beta,\alpha} \prod_{i<j} \left|\lambda_i - \lambda_j\right|^\beta \prod_i \lambda_i^{a-p} e^{-\sum_{i=1}^n \frac{\lambda_i}{2}} \text{ with } a = \frac{\beta}{2} n \text{ and } p = 1 + \frac{\beta}{2}(m-1)$$

$$c_L^{\beta,\alpha} = 2^{-ma} \prod_{j=1}^m \frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}j\right) \Gamma\left(a - \frac{\beta}{2}(m-j)\right)}$$

- **β-MANOVA/Jacobi case:**

$$f_\beta(\lambda) = c_L^{\beta,a_1,a_2} \prod_{i<j} \left|\lambda_i - \lambda_j\right|^\beta \prod_{j=1}^n \lambda_i^{a_1-p} (1-\lambda_i)^{a_2-p} \text{ with } a_1 = \frac{\beta}{2} n_1,\ a_2 = \frac{\beta}{2} n_2, \text{ and } p = 1 + \frac{\beta}{2}(m-1)$$

$$c_L^{\beta,a_1,a_2} = \prod_{j=1}^m \frac{\Gamma\left(1+\frac{\beta}{2}\right) \Gamma\left(a_1 + a_2 - \frac{\beta}{2})m - j)\right)}{\Gamma\left(1+\frac{\beta}{2}j\right) \Gamma\left(a_1 - \frac{\beta}{2}(m-j)\right) \Gamma\left(a_2 - \frac{\beta}{2}(m-j)\right)}$$

Tridiagonalization of the general β-ensembles

Hermite Matrix $n \in \mathbb{N}$	$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$
Laguerre Matrix $m \in \mathbb{N}$ $a \in \mathbb{R}$ $a > \frac{\beta}{2}(m-1)$	$L_\beta = B_\beta B_\beta^T$, where $B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{pmatrix}$

Tridiagonalization of the general β-ensembles

Tridiagonalizing the β-Gaussian/Hermite ensembles:

- Theorem 1 (Dumitriu and Edelman): Let $H_\beta = Q\Lambda Q^T$ be the eigendecomposition of H_β; fix the signs of the first row of Q to be non-negative and order the eigenvalues in increasing order on the diagonal of $\lambda = diag(\Lambda)$. Then λ and q, the first row of Q, are independent. Furthermore, the joint density of the eigenvalues is

$$f_\beta(\lambda) = c_H^\beta \prod_{i<j} \left|\lambda_i - \lambda_j\right|^\beta e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} = c_H^\beta |\Delta(\lambda)|^2 e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}}$$

- and $q = (q_1, \dots, q_n)$ is distributed as $(\chi_\beta, \dots, \chi_\beta)$, normalized to unit length.
- Here, the Vandermonde determinant for the ordered eigenvalues of a symmetric tridiagonal matrix with positive sub-diagonal $b = (b_{n-1}, \dots, b_1)$ is given by

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \frac{\prod_{i=1}^{n-1} b_i^i}{\prod_{i=1}^n q_i}$$

- where $(q_1, \dots, q_n)$ is the first row of the eigenvector matrix.

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

Tridiagonalization of the general β-ensembles

Tridiagonalizing the β-Laguerre/Wishart ensembles:

- Let $L_\beta = B_\beta B_\beta^T$ be the corresponding tridiagonal matrix.

- Theorem 2 (Dumitriu and Edelman): Let $L_\beta = Q\Lambda Q^T$ be the eigendecomposition of L_β; fix the signs of the first row of Q to be non-negative and order the eigenvalues increasingly on the diagonal of Λ. Then Λ and the first row of Q are independent. Furthermore, the joint density of the eigenvalues is

$$f_\beta(\lambda) = c_L^{\beta,\alpha} |\Delta(\lambda)|^\beta \prod_i \lambda_i^{a-p} \, e^{-\sum_{i=1}^n \frac{\lambda_i}{2}}$$

where $p = 1 + \frac{\beta}{2}(m-1)$, and q is distributed as $(\chi_\beta, \dots, \chi_\beta)$, normalized to unit length.

- Theorem 3 (Dumitriu and Edelman): Let G be an $m \times n$ matrix of i.i.d. standard Gaussians; then $W = GG^T$ is a Wishart real matrix. By reducing G to bidiagonal form B one obtains that the matrix $T = BB^T$ from the 1-Laguerre ensemble of Laguerre parameter $a = \frac{n}{2}$ has the joint eigenvalue p.d.f. of a Wishart ensemble.

$$B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{pmatrix}$$

Open issues in the study of general β-ensembles

1. *Jacobi (MANOVA) Ensembles*: The natural question is whether such models exist for the last member of the classical triplet, Jacobi. The Jacobi ensembles have been intensively studied as theoretical distributions, especially in connection with Selberg-type integrals and Jack (or Jack-Selberg) polynomials. Finding a random matrix model that corresponds to them would be of much interest.
2. *Level densities:* The level density of an ensemble is the distribution of a random eigenvalue of that ensemble (and by the Wigner semicircular law we know that the limiting distribution as $n \to \infty$ of such an eigenvalue is semicircular). The three functions found to be the level densities of the Gaussian models depend on the univariate Hermite polynomials. Finding a unified formula for the general case would be of interest.
3. *Level spacings.* The level spacings are the distances between the eigenvalues of an ensemble, usually normalized so that the average consecutive spacing is 1. These spacings have been well-studied in the case of the Gaussian ensembles (β = 1, 2, 4). The limiting probability density of a random spacing in these cases is known in terms of spheroidal functions. The level spacing of the general β -Hermite ensembles has not been investigated.
4. *Bulk and edge scaling limits.* Finally, a very important application would be the generalization of the bulk and edge scaling limits for the GOE, GUE and GSE obtained by Tracy and Widom.

(„Regular“ and generalized) multivariate Fuss-Narayana polynomials

- For any given $p \in N$ and any $n \in N$, consider the product of independent rectangular Gaussian random matrices:

$$B(n) = X_1(n)X_2(n) \dots X_p(n)$$

- Lenczewski (2012) shows that under certain natural assumptions

$$\lim_{n\to\infty} \tau_0(n)\left(\left(B(n)B^*(n)\right)^k\right) = P_k(d_0, d_1, \dots, d_p)$$

- where P_k is a homogenous polynomial of order $2pk$ for any $k \in \mathbb{N}$.

- Non-homogenous polynomials obtained from P_k by dividing them by d_0^{kp} are called **generalizations of Narayana polynomials**. In the case when $p > 1$ and all matrices in the product are square, the limit moments are Fuss-Narayana polynomials of one variable and they correspond in turn to the Fuss-Narayana numbers and their decompositions in terms of those numbers. For arbitrary asymptotic dimensions, it is therefore natural to expect that we should obtain some multivariate analogs of Fuss-Narayana polynomials.

(„Regular“ and generalized) multivariate Fuss-Narayana polynomials

- **Combinatorial definition of the polynomials $\boldsymbol{P_k}$:**

$$P_k(d_0, d_1, \dots, d_p) = \sum_{j_0+\dots+j_p=pk+1} \frac{1}{k}\binom{k}{j_0}\binom{k}{j_1}\dots\binom{k}{j_p} d_0^{j_0-1} d_1^{j_1} \dots d_p^{j_p}$$

- where the indices $j_0, j_1, \dots, j_p$ are natural numbers.

- Dividing by d_0^{kp} we obtain

$$F_k(t_1, t_2, \dots, t_p) = d_0^{-kp} P_k(d_0, d_1, \dots, d_p)$$

- where $t_j = d_j/d_0$ for any j, called multivariate Fuss-Narayana polynomials since for $p = 1$ the become regular Narayana polynomials. The coefficients above play the role of generalized Fuss-Narayana numbers.

- **Generalized Fuss-Narayana numbers:**

- For any $p, k \in \mathbb{N}$, the following decomposition holds:

$$C_k = \sum_{j_0+\dots+j_p=pk+1} N(k, j_0, \dots, j_p)$$

- where the summation runs over all $j_0, \dots, j_p \in [k] = \{1,2, \dots, k\}$, for which it holds that $j_0 + \dots + j_p = pk + 1$ and for such values

$$N(k, j_0, \dots, j_p) = \frac{1}{k}\binom{k}{j_0}\binom{k}{j_1}\dots\binom{k}{j_p}$$

(„Regular“ and generalized) multivariate Fuss-Narayana polynomials

- Non-homogenous polynomials of p variables of the form

$$F_k(t_1, t_2, \dots, t_p) = \sum_{j_0+\cdots+j_p=pk+1} N(k, j_0, \dots, j_p) t_1^{j_1} t_2^{j_2} \dots t_p^{j_p}$$

- where $k \in \mathbb{N}$, are called **multivariate Fuss-Narayana polynomials**.

- Moments of free multiplicative convolutions

$$\varrho_{t_1} \boxtimes \varrho_{t_2} \boxtimes \cdots \boxtimes \varrho_{t_p}$$

of Marchenko-Pastur laws with different shape parameters $t_1, t_2, \dots, t_p$ are multivariate Fuss-Narayana polynomials.

- Corollary (Lenczewski and Salapata, 2012): For any $k, p \in \mathbb{N}$, it holds that

$$F_k(t_1, t_2, \dots, t_p) = P_k(1, t_1, \dots, t_p)$$

- Proposition (Lenczewski and Salapata, 2012): For any positive $t_1, t_2, \dots, t_p$ and $k \in \mathbb{N}$, it holds that

$$m_k\left(\varrho_{t_1} \boxtimes \varrho_{t_2} \boxtimes \cdots \boxtimes \varrho_{t_p}\right) = F_k(t_1, t_2, \dots, t_p)$$

(„Regular“ and generalized) multivariate Fuss-Narayana polynomials

- Generalized multivariate Fuss-Narayana polynomials:

$$P_{m,r}(d_1, d_2, \dots, d_{p+1}) = \sum_{j_1+\cdots+j_{p+1}=mp+r} \frac{1}{n}\binom{m}{j_1} \dots \binom{m}{j_{p+1}} d_1^{j_1} \dots d_{p+1}^{j_{p+1}}$$

- where $m, r \in \mathbb{N}$ and summation runs over nonnegavtive integers (for $r = 1$ we obtain multivariate Fuss-Narayana polynomials). Limit moments of the class of Wishart type products constructed from blocks of random matrices are linear combinations of these polynomials. The operatorial description consists in expressing asymptotic distributions in terms of mixed moments of variables that can be viewed as matricial counterparts of canonical noncommutative random variables.

- Operatorial realizations of the limit moments can be used in terms of matricial counterparts of canonical noncommutative random variables of Voiculescu. The formalism of matricial freeness can be viewed as freeness with respect to a family of scalar-valued states and was developed and applied previously to asymptotic distributions of blocks of random matrices

Constructing level densities for general β-ensembles

- In classical quantum mechanics, the statistical properties of energy levels can be described by the k-level correlation functions defined as:

$$R_{n\beta}^{k}(x_1, x_2, \dots, x_k) = \frac{n!}{(n-k)!} \int \dots \int P_{n\beta}(x_1, x_2, \dots, x_n) dx_{k+1} \dots dx_n$$

- where $P_{n\beta}(x_1, x_2, \dots, x_n) = c_{n\beta} \cdot \exp(-\beta H)$ is the joint probability density function of n eigenvalues $x_1, x_2, \dots, x_n$ of a $n \times n$ random matrix, $c_{n\beta}$ is the normalized constant. H is the Hamiltonian of the logarithmical interacting n particles system on a straight line.

- When $k = 1$, $R_{n\beta}^{1}(x)$ can be explained as the distribution density of energy levels which can be found near by x. The level density is defined by the limit of the 1-level correlation function $R_{n\beta}^{1}(x)$.

- How to determine the level density – example of GUE:

- Using the recurrence formula of Hermite polynomials and the uniform integrability of random variable sequence, by the obtained differential equation, it can be shown that

$$\lim_{m\to\infty} \int f\left(\frac{x}{2\sqrt{m}}\right) \frac{1}{m} \sum_{k=0}^{m-1} p_k^2(x)\, \varpi(x) dx = \mathbb{E}(f(\sqrt{X}Y)), \text{for all } f \in C_b(\mathbb{R})$$

- By the analogous technique, level densities of Laguerre and Jacobi unitary ensembles, respectively, can also be obtained.

Constructing level densities for general β-ensembles

- General (unitary ensemble, i.e. $\beta = 2$) case:

$$R_n^1(x) = \sum_{m=0}^{n-1} p_m^2(x)\ \varpi(x)$$

- where $p_m(x)$ is the m-th orthogonal polynomial associated with the normalized weight function $\varpi(x) = \bar{c} \cdot \exp(-2V(x))$, where $\bar{c}$ is the normalized constant.

- Possibly - potential can be harmonic, as in Eynard, Kimura and Ribault (2018):

$$\mathcal{Z} = \int_{M_N(\mathbb{C})} dM e^{-N\mathcal{V}(M,M^\dagger)}$$

- where the total potential $\mathcal{V}(x,\bar{x}) = Rx\bar{x} + V(x) + \bar{V}(\bar{x})$ includes a Gaussian term with a constant coefficient R, and a harmonic term $V(x) + \bar{V}(\bar{x})$ and it holds $\partial\bar{\partial}\mathcal{V}(x,\bar{x}) = R$

Constructing level densities for general β-ensembles

- In our case, we use Fuss-Narayana product-type polynomials instead of orthogonal (β=2) or skew-orthogonal (β=1 or β=4, see Ghosh, 2009) ones
- We denote by S_μ the associated S-transform defined by:

$$S_\mu(z) = \frac{z+1}{z}\psi_\mu^{-1}(z)$$

- where ψ_μ is the moment generating function without constant term of the form

$$\psi_\mu(z) = \sum_{k=1}^{\infty} m_k z^k$$

$$\psi_\mu(z) = \sum_{k=1}^{\infty}\sum_{j_1,j_2} N_k(j_1,j_2)\, d_1^{j_1} d_2^{j_2} z^k$$

Constructing level densities for general β-ensembles

- Theorem 1a: Moments of the asymptotic distribution μ of BB^* under τ_1 are given by the formula

$$m_k = d_1^{-1} \sum_{r=1}^{k} P_{k,r}(d_1, d_2, \dots, d_{p+1}) T_{k,r}(t_1, t_2, \dots, t_r)$$

- where

$$T_{k,r}(t_1, t_2, \dots, t_r) = \sum_{i_1 + \cdots + i_k = r-1} t_{i1} t_{i2} \dots t_{ik}$$

- for any $k, r \in \mathbb{N}$, where numbers t_k are coefficients of the T-transform $T_{\tilde{v}}(z) = \sum_{k=0}^{\infty} t_k z^k$.
- Theorem 2a:

$$m_k^n = \frac{1}{n \cdot (D_n)^{k/2}} \sum_{j=0}^{n-1} \sum_{i=0}^{\left[\frac{k}{2}\right]} \sum_{T \in \Lambda_k^i} T \varrho_{j,L^2(\varpi)} \boxtimes \varrho_{j,L^2(\varpi)}$$

- Under the appropriate growth conditions, it holds:

$$\sigma_n(x) \xrightarrow{\omega} \sigma(x), n \to \infty$$

Application: sampling general β-ensembles

- Olver et al. (2014) – The task of sampling invariant ensembles can be reduced to:

1. Construct the orthogonal polynomials associated to the weight w ($w(x) = e^{-V_\beta(x)}$)
2. Sample a determinantal point process defined through this sequence of orthogonal polynomials.

- The first task can be accomplished via either Stieljes procedure or using Riemann-Hilbert techniques while for the second task, the authors used previous literature.

Algorithm 1 Sample determinantal processes, adapted from [17, 28]

Input: Chebyshev interpolations of $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_{n-1})^\top$
Output: n UIE eigenvalues $\boldsymbol{r} = (r_1, \ldots, r_n)$
Initialize $\boldsymbol{q}_n(x) = \boldsymbol{\phi}(x)$
for $k = n, \ldots, 2$ **do**
  Obtain r_k by sampling the PDF $\frac{\boldsymbol{q}_k(x)^\top \boldsymbol{q}_k(x)}{k}$
  Let $\boldsymbol{f}_k = \boldsymbol{q}_k(r_k) \in \mathbb{R}^k$
  Let $Q_k = \text{null}(\boldsymbol{f}_k^\top)$ (so that $Q_k^\top \boldsymbol{f}_k = 0$, and $Q_k \in \mathbb{R}^{k \times k-1}$)
  Let $\boldsymbol{q}_{k-1}(x) = Q_k^\top \boldsymbol{q}_k(x)$
end for
Obtain r_1 by sampling the PDF $\boldsymbol{q}_1^\top \boldsymbol{q}_1$

Algorithm 2 Sample unitary invariant ensembles

Input: Chebyshev interpolations of $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_{n-1})^\top$
Output: $n \times n$ UIE matrix
Obtain n eigenvalues $\boldsymbol{r}$ by calling Algorithm 1
Sample unitary matrix V from the Haar distribution. E.g., orthogonalize a random $n \times n$ complex matrix whose entries are iid complex Gaussians [29].
Return $V \operatorname{diag}\{\boldsymbol{r}\} V^\star$

- Proof of the validity of the sampling procedure, i.e. the algorithm indeed samples $P_n(x_1, \ldots, x_n)dx = \frac{1}{\hat{Z}_n} e^{-\sum_{i=1}^n Q(x_i)} \prod_{i<j}(x_i - x_j)^2 \, dx$, is provided.

Application: sampling general β-ensembles

- Algorithm 1: Sampling Pfaffian processes for $\beta = 1,4$

Input: Chebyshev interpolations (based on solutions to appropriate RH problems) of $\phi = (\phi_0, \dots, \phi_{n-1})^T$ and $\psi = (\psi_0, \dots, \psi_{n-1})^T$

Output: n OIE eigenvalues $r = (r_1, \dots, r_n)$

Initialize $p_n(x) = \phi(x)$ and $q_n(x) = \psi(x)$

for $k = n, \dots, 2$ do

Obtain r_k by sampling the PDF $\frac{p_k(x)^T q_k(x)}{\hat{Z}_{k,\beta}}$

Let $f_k = p_k(r_k) \in \mathbb{R}^k$ and $g_k = q_k(r_k) \in \mathbb{R}^k$

Let $P_k = \mathrm{skew}({f_k}^T)$ and $Q_k = \mathrm{skew}({g_k}^T)$

Let $p_{k-1}(x) = {P_k}^T p_k(x) + {p_k}^T(x) P_k$ and $q_{k-1}(x) = {Q_k}^T q_k(x) + {q_k}^T(x) Q_k$

end for

Obtain r_1 by sampling the PDF ${p_1}^T q_1$

Correction in the case of general β-ensembles: using free Fock spaces instead of Lie groups and the skew transformation.

Application: sampling general β-ensembles

- Algorithm 2: Sampling orthogonal/symplectic invariant ensembles

Input: Chebyshev interpolations (based on solutions to appropriate RH problems) of $\phi = (\phi_0, \ldots, \phi_{n-1})^T$ and $\psi = (\psi_0, \ldots, \psi_{n-1})^T$

Output: $n \times n$ $(2n \times 2n)$ OIE/SIE matrix

Obtain n eigenvalues r by calling Algorithm 1

Sample orthogonal/symplectic matrix V from the appropriate Haar distribution.

Return $V\mathrm{diag}\{r\}V^T$ / $V\mathrm{diag}\{r\}V^*$

Extensions and conclusion

- Derivation of the level densities formula for the general β-ensembles case (the general β-ensembles of Dumitriu and Edelman type, tridiagonalization), by exploiting the product nature of tridiagonalization procedure and noncommutative/free probability
- Possible extensions:

- Natural one: general β-Jacobi ensembles
- Other definitions of general β-ensembles
- Frequently discussed special case of $\beta = 6$
- Perturbation invariability: almost immediate
- Applications to other matrix ensembles problems: e.g. limiting entropy issues in Mészaros, 2020
- Using free probability to solve other open issues in Dumitriu and Edelman type general β-ensembles (e.g. bulk and edge scaling limits; level spacing of the general β-Hermite ensembles)
- Applying other approaches to address the above topics (e.g. Stein based asymptotic novelties – fourth moment based Malliavin calculus).

THANK YOU FOR LISTENING!

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