

NON-UNIFORMLY HYPERBOLIC DOMAINS FOR SOME CLASSES OF PIECEWISE SMOOTH SYSTEMS

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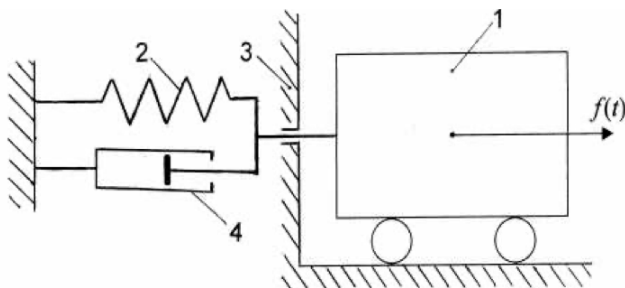
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A model of a system with an elastic impact

Reference

S. Kryzhevich, *Chaotic regimes of the systems, described by the Lienard equation with a big period of the right hand side and by the impact conditions*, Journal of Applied Mathematics and Mechanics **74** (2010), no.5, 751 – 773.



A model of the system: equations

Equation of motion between impacts

$$\ddot{x} + p(x)\dot{x} + q(x) = f(t/T). \quad (1)$$

where $t, x \in \mathbb{R}$, T is a big positive parameter.

Assumptions on the right-hand side f :

- f is a 1-periodic C^1 analytic function;
- f has exactly two roots over the period: 0 and τ_1 with $f'(0) > 0$, $f'(\tau_1) < 0$.

Let $T_1 = T\tau_1$, $T_2 = T - T_1$.

Assumptions on the left-hand side:

- p and q C^1 smooth functions that are analytic in a neighbourhood of $x = 0$;
-

$$p(x) \geq p_0 > 0; \quad q(0) = 0; \quad \omega^2(x) = q'(x) - p^2(x)/4 > 0 \quad (2)$$

for all x such that $q(x) \leq \max |f|$. W.l.g. $\omega(0) = 1$

A model of the system: elastic impact conditions

Newtonian elastic impact condition:

$$\dot{x}(t_0 + 0) = -\dot{x}(t_0 - 0) \quad \text{if } x(t_0) = 0. \quad (3)$$

Non-penetration condition

If $t_0 \in [Tn + T_2, T(n + 1)]$, $n \in \mathbb{Z}$ (or, in other words $f(t_0) < 0$) and $x(t_0) = \dot{x}(t_0) = 0$ then $x(t) \equiv 0$ on $[Tn + T_2, T(n + 1)]$.

Basic properties of solutions

Existence and uniqueness

Here we need to assume that the functions are analytic, see Paoli and Schatzmann, 2002;

Dissipativity

$$V = \left(\dot{x} + \int_0^x p(s)/2 ds \right)^2 + 2 \int_0^x q(s) ds;$$

$$h(t) := q^{-1}(f(t)).$$

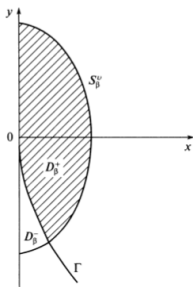
Lemma 1.

Equation (1) without impact condition has a unique T -periodic solution $\varphi(t)$ with

$$|\varphi(t) - h(t)| = O(1/T), \quad |\dot{\varphi}(t) - \dot{h}(t)| = O(1/T^2), \quad |\ddot{\varphi}(t) - \ddot{h}(t)| = O(1/T^3).$$

Moreover, $\varphi(t)$ has two zeroes over the period $t = \theta_0 \in (-C, C)$ and $t = \theta_1 \in (T_1 - C, T_1 + C)$.

The curve Γ



There exists a C^1 smooth curve $\Gamma: x = \gamma(y)$, $y \leq 0$ with $\gamma(0) = 0$ and such that in a neighbourhood of the origin:

- $$\gamma(y) = \sqrt{\frac{-8y^3}{9f'(0)}}(1 + o(1));$$

- If $x_0 < \gamma(y_0)$, the solution, starting at (x_0, y_0) for $t = 0$, has an impact on $[0, C]$; if $x_0 > \gamma(y_0)$, it does not on $[0, T_1]$.

Behaviour of solutions on $[0, T_1]$

Lemma 2.

Let r and θ be polar coordinates centered at $\varphi(t)$. Then any solution of Eq. (1) starting at a C - neighborhood of $\varphi(0)$ with initial conditions r_0 and θ_0 satisfy the condition

$$r(t) = r_0 \exp(-Y + o(T)), \quad \theta = \theta_0 - \int_0^t \omega(h(s)) ds + O(1/T)$$

where

$$Y = \frac{1}{2} \int_0^t p(h(s)) ds$$

One can imagine a slowly moving point of focus type.

A single low-velocity impact (grazing)

The matrix of the impact with the velocity y_k

$$\begin{pmatrix} -1 & 0 \\ -\zeta_k & -1 \end{pmatrix};$$

$$\zeta_k = 2f(t_k)/y_k + O(1).$$

'Overtaking' phenomenon. Experiment with ping-pong balls.

An now take into account motion between impacts

$$\begin{aligned} B_k &= \frac{\partial(x(t_{k+1} - 0), y(t_{k+1} - 0))}{\partial(x(t_k - 0), y(t_k - 0))}(0, y_k) = \\ &= \begin{pmatrix} -1 - \zeta_k \Delta_k + O(\Delta_k) & -\Delta_k + O(\Delta_k^2) \\ -\zeta_k + O(1) & -1 + O(\Delta_k) \end{pmatrix} \end{aligned}$$

Here $\Delta_k = t_{k+1} - t_k$.

The overall behaviour of solutions on $[T_1, T]$

Lemma 3

Any solution that is not zero has finitely many impacts on $[T_1, T]$ (incomplete chatter) with velocities y_1, y_2, \dots, y_N ($t = t_1, \dots, t_N$) such that

$$y_k / y_1 \in (\exp(-\Sigma_+(t_k)), \exp(-\Sigma_-(t_k))),$$

$$\Sigma_{\pm}(t) = \frac{1}{2}p(0)(t_k - t_1) - \frac{1}{3} \ln \left(\frac{f(t_{k-1})}{f(t_1)} \right) + \varepsilon(t_k)$$

where $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$.

Number of impacts increase exponentially with T .

The overall behaviour of solutions on $[T_1, T]$

$$B = \prod_{k=1}^N B_k \text{ Let } S(x(0), y(0)) = (x(T), y(T))$$

The Jacobi matrix

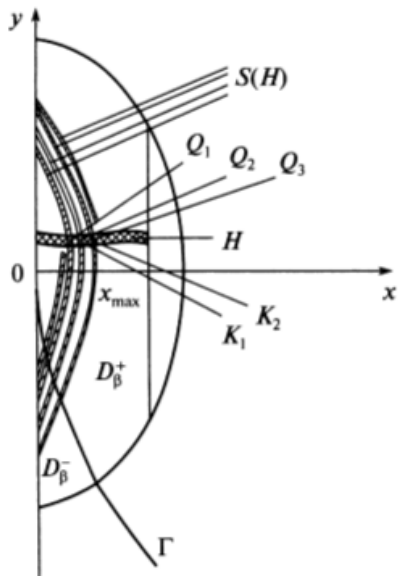
$$\frac{\partial(x(T), y(T))}{\partial(x(0), y(0))} = D = AB, \quad A = \exp(-Y(1 + O(1/T))R_\theta(T_1))$$

$$\det D \sim \exp(-Y - p(0)T_2), \quad \text{Tr } D \sim \exp(\Sigma_\pm)$$

$$\lambda_+ \sim \exp(\Sigma_\pm), \quad \lambda_- \sim \exp(-\Sigma_\pm)$$

Both directions of eigenvectors tend to vertical.

Smale horseshoe and non-uniform hyperbolicity



Main results

Theorem

Suppose the coefficients of Eq. (1) are such that conditions (2) are satisfied.

Then, for a given system (1)+(3) there exists a number $\bar{T} > 0$ such that for any $T > \bar{T}$, the T -shift mapping S , corresponding to the system (1)+(3), has a non-trivial compact hyperbolic transitive invariant set K where periodic points are dense.

The mapping $S|_K$ is conjugated to the Bernoulli shift.

Conjecture

Besides, the map S admits infinitely many sinks and the set K is a subset of the boundary of the trapping region for these sinks.

Incomplete chatter, what else?

- A function f having more than 2 zeroes;
- $\dim x > 1$;
- other systems with impulse type conditions e.g. systems with Coulomb friction;
- systems with hysteresis;
- van der Pol equation???

Thank you! Hvala lepo!