SHOULD I STAY OR SHOULD I GO? ZERO-SIZE JUMPS IN RANDOM WALKS FOR LÉVY FLIGHTS

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Gianni Pagnini

BCAM
Alameda de Mazarredo 14
48009 Bilbao
Basque Country – Spain
gpagnini@bcamath.org

In collaboration with S. Vitali

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basque center for applied mathematics

ikerbasque
Basque Foundation for Science

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EXCELENCIA SEVERO OCHOA

GOBIERNO DE ESPAÑA MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD

Motivation

This research is motivated by the fact that, in the literature dedicated to random walks for anomalous diffusion,

it is disregarded if

the walker does not move in the majority of the iterations because the most frequent jump-size is zero, namely *"Should I stay?"*:

the jump-size distribution is unimodal with mode in zero,

or, in opposition, if the walker always moves because the jumps with zero-size never occur, namely *"Should I go?"*:

the jump-size distribution is bi-modal and null in zero.

Prequel

Let $\rho(\boldsymbol{x};t)$ be the distribution of the walker's displacement $\boldsymbol{x} \in \mathbb{R}^N$ at time $t > 0$ and let $\varphi(\Delta\boldsymbol{x})$ be the jump distribution, then

$$\rho(\boldsymbol{x}; t+\Delta t) = \int_{\mathbb{R}^N} \rho(\boldsymbol{x}-\Delta\boldsymbol{x}; t)\, \varphi(\Delta\boldsymbol{x})\, d\Delta\boldsymbol{x}\,, \tag{1}$$

and, by using the Taylor expansion of $\rho(\boldsymbol{x};t)$ together with the symmetry of $\varphi(\Delta\boldsymbol{x})$, the diffusion equation is obtained:

$$\frac{\partial \rho}{\partial t} = \mathcal{D}\,\Delta\rho\,, \quad \mathcal{D} = \frac{\langle(\Delta\boldsymbol{x})^2\rangle}{2\Delta t}\,. \tag{2}$$

"Should I stay?": $\varphi(0) = \sup\{\varphi(\Delta\boldsymbol{x}) : \Delta\boldsymbol{x} \in \mathbb{R}^N\}$,

"Should I go?": $\varphi(0) = \inf\{\varphi(\Delta\boldsymbol{x}) : \Delta\boldsymbol{x} \in \mathbb{R}^N\} = 0$.

What about when $\langle(\Delta x)^2\rangle \to \infty$?

Formula (1), in the lattice $h\mathbb{Z}^N$ reads

$$\rho(\boldsymbol{x}; t + \Delta t) = \sum_{\boldsymbol{z}\in\mathbb{Z}^N} \varphi(h\boldsymbol{z})\rho(\boldsymbol{x} - h\boldsymbol{z}; t)h^N .$$

If $\varphi(\Delta\boldsymbol{x}) = |\Delta\boldsymbol{x}|^{-N-\alpha}$, with $\varphi(0) = 0$, and $\lim_{\substack{h\to 0\\ \Delta t\to 0}} \dfrac{h^\alpha}{\Delta t} \to \mathcal{D}_\alpha$,

$$\frac{\partial\rho}{\partial t} = -\mathcal{D}_\alpha \int_{\mathbb{R}^N} \frac{\rho(\boldsymbol{y}; t) - \rho(\boldsymbol{x}; t)}{|\boldsymbol{x} - \boldsymbol{y}|^{N+\alpha}} d\boldsymbol{y} = -\mathcal{D}_\alpha(-\Delta)^{\frac{\alpha}{2}}\rho\,, \quad 0 < \alpha < 2\,.$$

The distinctive singularity of the fractional Laplacian is mapped by the bi-modal shape of the distribution of jumps.

E. Valdinoci, From the long jump random walk to the fractional Laplacian. Bol. Soc. Esp. Mat. Apl. 49 (2009), 33–44.

What about when $\langle(\Delta x)^2\rangle \to \infty$?

The characteristic function proceedure

$$\rho(\boldsymbol{x}; t+\Delta t) = \int_{\mathbb{R}^N} \rho(\boldsymbol{x}-\Delta\boldsymbol{x}; t)\, \varphi(\Delta\boldsymbol{x})\, d\Delta\boldsymbol{x}\,,$$

$$\widehat{\rho}(\boldsymbol{\kappa}; t+\Delta t) = \widehat{\rho}(\boldsymbol{\kappa}; t)\widehat{\varphi}(\boldsymbol{\kappa})\,,$$

$$\widehat{\rho}(\boldsymbol{\kappa}; t+\Delta t) \simeq \widehat{\rho}(\boldsymbol{\kappa}; t) + \Delta t \frac{\partial \widehat{\rho}}{\partial t}\,,$$

$$\widehat{\varphi}(\boldsymbol{\kappa}) \simeq 1 - \ell^\alpha\, |\boldsymbol{\kappa}|^\alpha + o(|\boldsymbol{\kappa}|^\alpha)\,, \quad \ell|\boldsymbol{\kappa}| \ll 1\,, \quad 0 < \alpha \le 2\,,$$

$$\frac{\partial \rho}{\partial t} + \mathcal{D}_\alpha(-\Delta)^{\frac{\alpha}{2}}\rho = 0\,, \quad \mathcal{D}_\alpha = \frac{\ell^\alpha}{\Delta t}\,. \qquad (3)$$

Markovian Continuous Time Random Walk

Let $\psi(t)$ be the waiting-times distribution and $\widetilde{\psi}(s)$ its Laplace transform, then (Montroll–Weiss, 1965)

$$\widehat{\widetilde{\rho}}(\boldsymbol{\kappa}, s) = \frac{1 - \widetilde{\psi}(s)}{s\,[1 - \widehat{\varphi}(\boldsymbol{\kappa})\widetilde{\psi}(s)]} \,,$$

and the process is Markovian if $\psi(t) = \mathrm{e}^{-t/\tau}/\tau$, therefore

$$\widehat{\rho}(\boldsymbol{\kappa}; t) = \mathrm{e}^{-(1-\widehat{\varphi}(\boldsymbol{\kappa}))t/\tau} \,, \quad \boldsymbol{\kappa} \in \mathbb{R}^N \,, \tag{4}$$

and, when $\widehat{\varphi}(\boldsymbol{\kappa}) \simeq 1 - \ell^\alpha |\boldsymbol{\kappa}|^\alpha + o(|\boldsymbol{\kappa}|^\alpha)$, with $0 < \alpha \leq 2$, it holds

$$\frac{\partial \rho}{\partial t} + \mathcal{D}_\alpha (-\Delta)^{\frac{\alpha}{2}} \rho = 0 \,, \quad \mathcal{D}_\alpha = \frac{\ell^\alpha}{\tau} \,.$$

The Brownian motion ($\alpha = 2$)

"Should I stay?"

$$\varphi(\boldsymbol{x}) = \frac{1}{(4\pi\ell^2)^{N/2}}\, \mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{4\ell^2}}\,, \tag{5}$$

$$\widehat{\varphi}(\boldsymbol{\kappa}) = \mathrm{e}^{-\ell^2|\boldsymbol{\kappa}|^2} \simeq 1 - \ell^2|\boldsymbol{\kappa}|^2 + \frac{\ell^4}{2}|\boldsymbol{\kappa}|^4 + o(|\boldsymbol{\kappa}|^4)\,, \quad \ell|\boldsymbol{\kappa}| \ll 1\,.$$

"Should I go?"

$$\varphi(\boldsymbol{x}) = \frac{1}{2}[\delta(\boldsymbol{x} - \sqrt{2}\ell\,\widehat{\boldsymbol{e}}) + \delta(\boldsymbol{x} + \sqrt{2}\ell\,\widehat{\boldsymbol{e}})]\,, \tag{6}$$

$$\widehat{\varphi}(\boldsymbol{\kappa}) = \cos(\sqrt{2}\ell\,\boldsymbol{\kappa}\cdot\widehat{\boldsymbol{e}}) \simeq 1 - \ell^2\,|\boldsymbol{\kappa}|^2 + \frac{\ell^4}{6}|\boldsymbol{\kappa}|^4 + o(|\boldsymbol{\kappa}|^4)\,, \quad \ell|\boldsymbol{\kappa}| \ll 1\,.$$

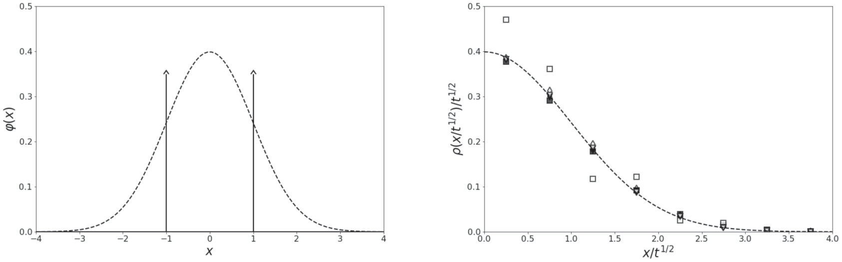

Figure: Left: plots of the one-dimensional ($N = 1$) jump *pdfs* (5) and (6) corresponding to the *Should I stay?* and *Should I go?* conditions, respectively, for the generation of the Brownian motion from CTRW models. Right: plots of the Gaussian walker's distribution $\rho(\boldsymbol{x}; t)$ (4) of the CTRW models for the Brownian motion as generated by using the one-dimensional ($N = 1$) jump *pdfs* (5) (filled symbols) and (6) (empty symbols) at $t = 10\tau$, 100τ , 1000τ represented by squares, diamons and triangles, respectively: the short-time effects of the coin-flipping rule (6) is visible.

CTRW models for Lévy flights ($0 < \alpha < 2$)

“Should I stay?"

$$\varphi(\boldsymbol{x}) = \frac{1}{\ell^N} \mathcal{L}_\alpha \left(\frac{\boldsymbol{x}}{\ell} \right) \sim \frac{1}{|\boldsymbol{x}|^{\alpha+N}} \,, \quad |\boldsymbol{x}| \to +\infty \,, \tag{7}$$

$$\widehat{\varphi}(\boldsymbol{\kappa}) = \mathrm{e}^{-\ell^\alpha \, |\boldsymbol{\kappa}|^\alpha} \simeq 1 - \ell^\alpha \, |\boldsymbol{\kappa}|^\alpha + \frac{\ell^{2\alpha}}{2} |\boldsymbol{\kappa}|^{2\alpha} + o(|\boldsymbol{\kappa}|^{2\alpha}) \,, \quad \ell |\boldsymbol{\kappa}| \ll 1 \,,$$

and $\rho(\boldsymbol{x}; t)$ solves (3).

CTRW models for Lévy flights

"Should I go?"

$$\varphi(x) = \begin{cases} \frac{1}{2} \frac{1}{\sqrt{2}\,\ell} \mathcal{L}_\alpha^{-\alpha}\left(\frac{|x|}{\sqrt{2}\ell}\right) \sim \frac{1}{|x|^{\alpha+1}}, & |x| \to +\infty, \qquad (8a) \\ \frac{1}{2} \frac{\alpha}{\Gamma(1/\alpha)|x|} \mathcal{L}_\alpha^{-\alpha}\left(\frac{|x|}{\sqrt{2}\ell}\right) \sim \frac{1}{|x|^{(\alpha+1)+1}}, & |x| \to +\infty. \qquad (8b) \end{cases}$$

Where $\mathcal{L}_\alpha^{-\alpha}(x)$, with $x \in \mathbb{R}$, is the one-sided (extremal) Lévy densities i.e., $\mathcal{L}_\alpha^{-\alpha}(x) > 0$ when $x > 0$ and $\mathcal{L}_\alpha^{-\alpha}(x) = 0$ when $x \leq 0$, with $0 < \alpha < 1$, and $\mathcal{L}_1^{-1}(x) = \delta(x-1)$. The power-law of the tails of the jump *pdf* $\varphi(x)$ spans inside the range of the stable parameter $(0,1) \cup (1,2)$.

From the Lévy coin-flipping rule (8a), for $\kappa \in \mathbb{R}$, we have that

$$\begin{aligned} \widehat{\varphi}(\kappa) &\simeq 1 - \sin\left[\frac{\pi}{2}(1+\alpha)\right](\sqrt{2}\ell\,|\kappa|)^{\alpha} \\ &\quad + \frac{1}{2}\sin\left[\frac{\pi}{2}(1+2\alpha)\right](\sqrt{2}\ell\,|\kappa|)^{2\alpha} + o(|\kappa|^{2\alpha})\,, \end{aligned} \tag{9}$$

hence expansion (9) is an alternating series if $0 < \alpha \leq 1/2$ such that $\rho(x;t)$ solves (3), but if $1/2 < \alpha < 1$ then we have

$$\sin\left[\frac{\pi}{2}(1+\alpha)\right] > 0 \quad \text{and} \quad \sin\left[\frac{\pi}{2}(1+2\alpha)\right] < 0\,, \tag{10}$$

$$\widehat{\varphi}(\kappa) \simeq 1 - \ell_\alpha|\kappa|^\alpha - \frac{\ell_{2\alpha}}{2}|\kappa|^{2\alpha} + o(|\kappa|^{2\alpha})\,. \tag{11}$$

and therefore (9) is not an alternating series, and $\rho(x;t)$ solves the fractional evolution problem

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathcal{K}_{\alpha}\,(-\Delta)^{\frac{\alpha}{2}}\rho + \dfrac{1}{2}\mathcal{K}_{2\alpha}\,(-\Delta)^{\alpha}\rho = 0\,, \quad \text{in } \mathbb{R}\times(0,+\infty)\,, \\ \rho(x;0) = \delta(x)\,, \\ \dfrac{1}{2} < \alpha < 1\,, \end{cases} \tag{12}$$

where

$$\mathcal{K}_{\alpha} = \frac{\ell_{\alpha}}{\tau} = \mathcal{D}_{\alpha}\,2^{\alpha/2}\left|\sin\left[\frac{\pi}{2}(1+\alpha)\right]\right|\,, \quad \mathcal{K}_1 = 0\,,$$

$$\mathcal{K}_{2\alpha} = \frac{\ell_{2\alpha}}{\tau} = \mathcal{D}_{2\alpha}\,2^{2\alpha/2}\left|\sin\left[\frac{\pi}{2}(1+2\alpha)\right]\right|\,, \quad \frac{1}{2}\mathcal{K}_2 = \mathcal{D}_2 = \mathcal{D}\,,$$

such that $\rho(x;t)$ is a convolution of Lévy stable densities:

$$\begin{aligned}\rho(x;t) &= \int_{\mathbb{R}^n} \mathcal{L}_\alpha(x-\xi;t)\mathcal{L}_{2\alpha}(\xi;t)\,d\xi \\ &= \frac{1}{(\mathcal{K}_\alpha\sqrt{\mathcal{K}_{2\alpha}/2}\,t^{3/2})^{N/\alpha}}\int_{\mathbb{R}^n}\mathcal{L}_\alpha\left(\frac{x-\xi}{(\mathcal{K}_\alpha t)^{1/\alpha}};1\right) \\ &\qquad \times\mathcal{L}_{2\alpha}\left(\frac{\xi}{(\mathcal{K}_{2\alpha}t/2)^{1/(2\alpha)}};1\right)\,d\xi\,,\end{aligned} \tag{13}$$

with fractional absolute moments

$$\sigma^q \propto \begin{cases} (\mathcal{K}_{2\alpha}t)^{q/(2\alpha)}\,, & t\to 0\,, \\ (\mathcal{K}_\alpha t)^{q/\alpha}\,, & t\to+\infty\,, \end{cases} \qquad 0<q<\alpha\,. \tag{14}$$

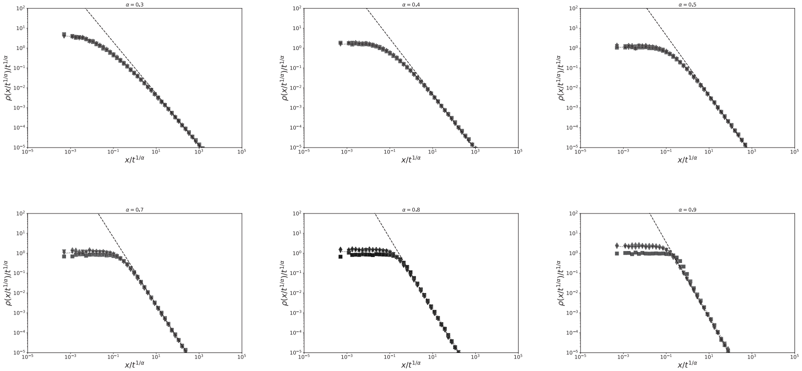

Figure: Plots of the central part of the walker's distribution $\rho(x; t)$ obtained with jump *pdf* (8a) at times $t = 10\tau, 100\tau, 1000\tau$ marked by squares, triangles and diamonds, respectively. The dotted lines represent Lévy stable densities of index α and the dashed lines the power-law decaying $|x|^{-(\alpha+1)}$. The loss of self-similarity in the interval $1/2 < \alpha < 1$ is evident.

From the Lévy coin-flipping rule (8b), we have that

$$\widehat{\varphi}(\kappa) = 1 - \frac{\alpha}{\Gamma(1/\alpha)} \frac{\sin(\pi\alpha/2)}{1+\alpha} (\sqrt{2}\ell\,|\kappa|)^{\alpha+1} + \frac{\alpha}{\Gamma(1/\alpha)} \frac{\sin(\pi\alpha)}{1+2\alpha} (\sqrt{2}\ell\,|\kappa|)^{2\alpha+1} + o(|\kappa|^{2\alpha+1}), \quad (15)$$

since $0 < \alpha < 1$, expansion (15) is an alternating series and $\rho(x;t)$ solves (3) by replacing $\alpha \to (\alpha+1)$ and

$$\mathcal{D}_\alpha \to \mathcal{D}_\alpha = \frac{2^{(\alpha+1)/2}}{\Gamma(1/\alpha)} \frac{\alpha}{1+\alpha} \sin\left[\frac{\pi}{2}\alpha\right] \frac{\ell^{\alpha+1}}{\tau}.$$

Indetermined homecoming

Diffusive processes, whose walker's distribution solves the fractional diffusion equation (3) with $0 < \alpha < 2$, are always transient except in the one-dimensional ($N = 1$) case when $1 \leq \alpha < 2$:

$$Q(r,t) = 1 - \frac{1}{t^{N/\alpha}} \int_{B_r} \rho\left(\frac{\mathbf{x}}{t^{1/\alpha}}; 1\right) d\mathbf{x} \in \left[1 - \frac{\mu |B_r|}{t^{N/\alpha}}, 1 - \frac{\nu |B_r|}{t^{N/\alpha}}\right],$$

$$\begin{cases} Q(0) = 0\,, \quad N = 1 \text{ with } 1 \leq \alpha < 2 \quad \text{(recurrent)}\,, \\ \\ Q(0) = 1\,, \quad N \geq 2\,, \text{ and } N = 1 \text{ with } 0 < \alpha < 1 \quad \text{(transient)}\,. \end{cases}$$

E. Affili, S. Dipierro, E. Valdinoci, Decay estimates in time for classical and anomalous diffusion. In: 2018 MATRIX Annals, MATRIX Book Series, Vol. 3, Springer, Cham (2020), 167–182.

In the considered case, when $\ell|\kappa| \ll 1$ and $t/\tau \to \infty$, the expected scaling

$$\rho(0;t) \sim \frac{\Gamma(1/\alpha)}{\alpha\pi\mathcal{K}_\alpha^{1/\alpha}}\, t^{-1/\alpha}\,, \quad 1/2 < \alpha < 1\,, \quad t \to \infty\,, \qquad (17)$$

is attained when the following large-time limit is reached

$$t \gg T = \frac{\tau}{\alpha^2}\frac{\left|\sin\left[\frac{\pi}{2}(1+2\alpha)\right]\right|}{\left[\sin\left[\frac{\pi}{2}(1+\alpha)\right]\right]^2}\,, \qquad (18)$$

but $T \to \infty$ when $\alpha \to 1$, that poses an issue for real systems.

In the transient regime $\tau \ll t \ll T$, from simulations it emerges

$$\rho(0;t) \sim t^{-1/[\alpha+f(\alpha)]}\,, \quad f_e(\alpha) = \frac{1}{\alpha^2}\frac{\Gamma(2\pi\alpha)-\Gamma(\pi)}{\Gamma(2\pi)-\Gamma(\pi)}\,. \qquad (19)$$

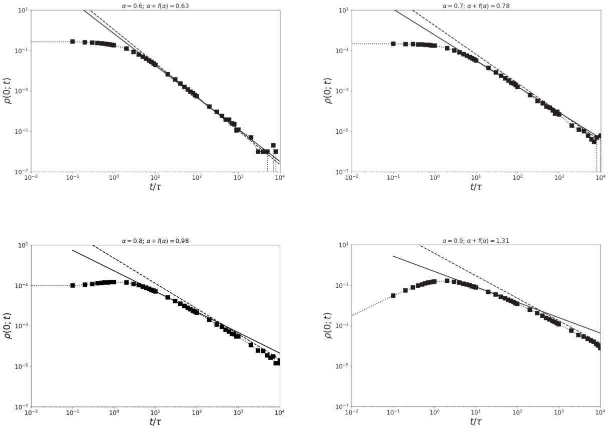

Figure: Plots of the decreasing in time of the maximum of the walker's distribution $\rho(0; t)$ generated through the jump *pdf* (8a) with $\alpha = 0.6\,, 0.7\,, 0.8\,, 0.9$. The solid line represent the decaying-law $t^{-1/[\alpha+f(\alpha)]}$ (19) and the dashed line is the large-time decaying-law $t^{-1/\alpha}$ (17). The plots show the duration of the intermediate regime $\tau \ll t \ll T$ and its enlarging as $\alpha \to 1$.

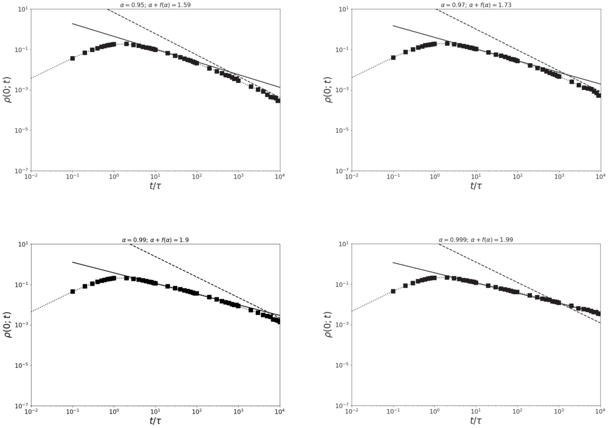

Figure: The same as in Figure 3 but with $\alpha = 0.95\,, 0.97\,, 0.99\,, 0.999$ for highlithing the delay in attaining the large-time limit $t \gg T$.

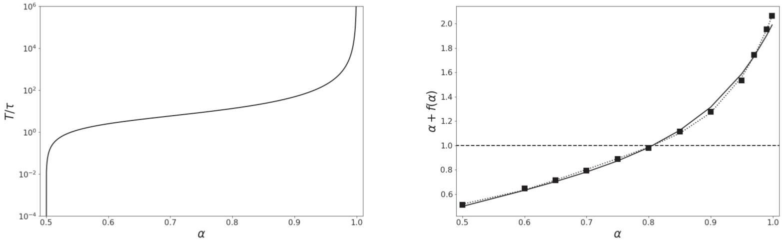

Figure: Left: Plot of the time-scale T as defined in formula (18). Right: Plot of the decaying-law of $\rho(0; t)$ as estimated by simulations (black squares). The dotted line corresponds to the formula $\alpha + c_2\alpha^2 + \cdots + c_6\alpha^6$ as provided by the fitting routine scipy.optimize.curve_fit while the solid line corresponds to the formula $\alpha + f(\alpha)$ (19) and the dashed line is the reference-line indicating the transient-to-recurrence conversion at $\alpha + f(\alpha) = 1$.

Summary and conclusions

- The characteristic function procedure does not catch the peculiar role of the *"Should I go?"* condition that is mapped into the distinctive singularity of the fractional Laplacian.

- When the small wavenumber expansion of the characteristic function of the jump *pdf* is not an alternating series then the process is not self-similar.

- The loss of self-similarity introduces a time-scale T that depends on α and tends to infinity when $\alpha \to 1$ and this makes the large-time limit unattainable in real systems.

- The long-extended intermediate regime $\tau \ll t \ll T$ could display a recurrence-like scaling, in spite of the transient theoretical one, that leads to an indetermined situation in real cases.

- If animal movement is modelled through Lévy-like motions then the serching for food, and also the searching for home, can be affected by the adopted jump rule: the searching for food could lead to a double-order fractional equation and the searching for home to an indetermined homecoming in real systems.

G. Pagnini, S. Vitali, Should I stay or should I go? Zero-size jumps in random walks for Lévy flights. Fract. Calc. Appl. Anal. **24**(1), 137–167 (2021); arXiv:2103.06981

Extra

We report here the main steps related to the calculations concerning the jump *pdf*s (8a) and (8b) providing the Lévy coin-flipping rules for the *"Should I go?"* condition.

Since the considered jump *pdf*s are symmetric, the corresponding characteristic functions are defined by

$$\widehat{\varphi}(\kappa) = 2 \int_0^\infty \cos(\kappa x) \varphi(x) dx \,, \tag{20}$$

that are symmetric as well, i.e., $\widehat{\varphi}(\kappa) = \widehat{\varphi}(-\kappa)$, and they can be expressed through their Mellin transform, i.e., with $x > 0$:

$$\varphi^*(s) = \int_0^\infty \varphi(x)\, x^{s-1} dx \,, \quad \varphi(x) = \frac{1}{2\pi i} \int_L \varphi^*(s)\, x^{-s} ds \,, \quad x > 0 \,.$$

The Mellin–Barnes integral representation turns out to be

$$\widehat{\varphi}(\kappa) = \frac{2}{\kappa}\frac{1}{2\pi i}\int_L \varphi^*(s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\kappa^s ds\,, \quad \kappa > 0\,, \qquad (21)$$

and, by reminding the Mellin–Barnes integral representation of extremal Lévy densities, i.e.,

$$\mathcal{L}_\alpha^{-\alpha}(x) = \frac{1}{\alpha}\frac{1}{2\pi i}\int_L \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(1-s)}\,x^{-s}ds\,,$$

and the related Mellin transform, i.e.,

$$\int_0^\infty \mathcal{L}_\alpha^{-\alpha}(x)\,x^{s-1}dx = \frac{1}{\alpha}\frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(1-s)} = \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(2-s)}\,,$$

for $\kappa > 0$, it results

$$\widehat{\varphi}(\kappa) = \frac{1}{\alpha\kappa} \frac{1}{2\pi i} \int_L \Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right) \sin\left(\frac{\pi s}{2}\right) \kappa^s ds\,, \qquad (22)$$

and by applying the residue theorem for $\kappa \to 0$ formula (9) is obtained, and analogously for the jump *pdf* (8b), for $\kappa > 0$, it holds

$$\widehat{\varphi}(\kappa) = \frac{1}{\Gamma(1/\alpha)\kappa} \frac{1}{2\pi i} \int_L \Gamma\left(\frac{2}{\alpha} - \frac{s}{\alpha}\right) \frac{\Gamma(1-s)}{\Gamma(2-s)} \sin\left(\frac{\pi s}{2}\right) \kappa^s ds\,, \qquad (23)$$

and by applying the residue theorem for $\kappa \to 0$ formula (15) is obtained.