Striped patterns for generalized antiferromagnetic functionals with power law kernels of exponent smaller than d + 2

8th ECM

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21.06.2021

Pattern formation

Spontanous formation of regular structures is observed in nature at different scales

- Biology, Material Science
- Experiments and Simulations
- Examples: polymers, colloidal suspensions, Langmuir monolayers, thin magnetic films, amphiphile solutions
- Applications: memory storage device; artificial photonic crystals; biological devices for separation of biomolecules on microchips, nanolithography

Pattern formation

Figure: Black-and-white striped zebra and polka-dot zebra [Caters News Agency Ltd.]

Generalized antiferromagnetic model in sharp interface version

Striped patterns are expected to emerge as minimizers/ground states of the following free-energy functional

$$\tilde{\mathcal{F}}_{\alpha,J,L}(E) = \frac{1}{L^d}\Big(\overbrace{J\,\mathrm{Per}_1(E,L)} - \underbrace{\int_{[0,L)^d}\int_{\mathbb{R}^d} K_{\alpha,1}(\zeta)|\chi_E(x)-\chi_E(y)|}\Big)$$

1. $J > 0$, $J \in [J_C - \tau, J_C)$ where $\tau > 0$ and $J_c := \int_{\mathbb{R}^d} |\zeta_1| K_{\alpha,1}(\zeta)\,\mathrm{d}\zeta$ is critical constant above which the minimizers are trivial
2. $\mathrm{Per}_1(E,[0,L)^d) := \int_{\partial E\cap[0,L)^d} \|\nu^E(x)\|_1 \,\mathrm{d}\mathcal{H}^{d-1}(x)$, where $\|y\|_1 = \sum_{i=1}^d |y_i|$ is the 1-norm
3. $K_{\alpha,1}$ is the power kernel, $K_{\alpha,1}(\zeta) = \frac{1}{(\|\zeta\|_1+1)^{p(\alpha)}}$, $\quad p(\alpha) = d+2-\alpha$

Variety of cases

- $p = d - 2 \quad (\alpha = 4)$ diblock copolymer
- $p = d \quad (\alpha = 2)$ micromagnetics
- $p = d + 1 \quad (\alpha = 1)$ thin magnetic films
- $p > d + 1 \quad (\alpha < 1)$ generalized antiferromagnetic model

Aim: to improve current results towards the physical cases

Rescaled model

Rescale the functional in such a way that optimal stripes width and energy are of order O(1)

$$\mathcal{F}_{\alpha,\tau,L}(E) := \begin{cases} \frac{1}{L^d}\Big[\mathrm{Per}_1(E;[0,L)^d)\Big(-1+\int_{\mathbb{R}^d} K_{\alpha,\tau}(\zeta)|\zeta_1|\,\mathrm{d}\zeta\Big) \\ \quad -\int_{\mathbb{R}^d}\int_{[0,L)^d} |\chi_E(x)-\chi_E(x+\zeta)|K_{\alpha,\tau}(\zeta)\,\mathrm{d}x\,\mathrm{d}\zeta\Big] \end{cases} \tag{1}$$

where

$$K_{\alpha,\tau}(\zeta) = \frac{1}{(\|\zeta\|_1+\tau^{1/(1-\alpha)})^{p(\alpha)}}. \tag{2}$$

Striped pattern formation $\alpha \leq 0, \quad p \geq d+2$

- Daneri and Runa '18 showed that, for $p \geq d+2$ (i.e. $\alpha \leq 0$) global minimizers of $\mathcal{F}_{\tau,L}$ are, when τ is sufficiently small, periodic unions of stripes. Moreover, their results hold in the large volume limit, namely for arbitrarily large L whenever L is an even multiple of h^*_τ, which is the optimal stripes' period among all possible periods.
- for the discrete analogue of the above functional and exponents $p > 2d$ a characterization of minimizers was first proved by Giuliani, Seiringer '16
- one-dimensionality and periodicity for a diffuse interface version of the model for exponents $\alpha \leq 0$ has been proved by Daneri, K., Runa '19 and Daneri, Runa '21 in the large volume limit

Striped pattern formation $0 < \alpha \leq \bar{\alpha} \ll 1$

Theorem (K. '21)

Let $d \geq 1$, $L > 0$. Then there exists $0 < \bar{\alpha} \ll 1$ and $\bar{\tau}_L > 0$ such that $\forall 0 < \alpha \leq \bar{\alpha}$ and $\forall 0 < \tau \leq \bar{\tau}_L$ the minimizers of the functional $\mathcal{F}_{\alpha,\tau,L}$ are periodic unions of stripes of width $h_{\tau,L,\alpha}$.

For $\tau, \alpha > 0$, let $h^*_{\tau,\alpha}$ be optimal among all widths of stripes for $\mathcal{F}_{\alpha,\tau,L}$ as L varies.

Theorem (K. '21)

There exists $0 < \bar{\alpha} \ll 1$ and $\bar{\tau} > 0$ such that $\forall 0 < \alpha \leq \bar{\alpha}$, $\forall 0 < \tau \leq \bar{\tau}$ and for all $k \in \mathbb{N}$, $L = 2kh^*_{\tau,\alpha}$, minimizers of $\mathcal{F}_{\alpha,\tau,L}$ are periodic stripes of width $h^*_{\tau,\alpha}$.

Idea for strategy of the proof

We know by the results of Daneri, Runa that minimizers for $\alpha \leq 0$ and τ small are stripes. Such a result is obtained by:

- for fixed $\alpha \leq 0$ let $\tau \to 0$, a rigidity estimate ensures that minimizers of the limit functional are stripes
- once close to stripes, for $\tau > 0$ show a stability result proving that minimizers are exactly stripes

Difficulties:

- for a fixed $\alpha > 0$, and $\tau \to 0$ no rigidity result is available, this result is sharp for $p = d + 2$, $(\alpha = 0)$

Instead:

- we consider $(\alpha, \tau) \downarrow (0, 0)$
- show Γ-convergence of $\mathcal{F}_{\alpha,\tau,L}$ to $\mathcal{F}_{0,0,L}$, where by the above $\mathcal{F}_{0,0,L}$ satisfies the rigidity estimate
- show a stability result for $0 < \alpha < 1$, $0 < \tau \ll 1$

Strategy of the proof

1. Decomposition of the functional penalizing derivations from one-dimensional profiles
2. Γ-convergence as $(\alpha, \tau) \downarrow (0, 0)$ to a functional which is finite only on stripes
3. Stability estimates for $0 < \alpha < 1$, $0 < \tau \ll 1$ showing that once close to stripes minimizers are exactly stripes
4. One-dimensional optimization through reflection positivity showing periodicity

Decomposition of the functional

$$\mathcal{F}_{\alpha,\tau,L}(E) \geq \frac{1}{L^d}\Big(-\sum_{i=1}^{d} \mathrm{Per}_{1i}(E,[0,L)^d)+\sum_{i=1}^{d} \mathcal{G}^i_{\alpha,\tau,L}(E)+\sum_{i=1}^{d} \mathcal{I}^i_{\alpha,\tau,L}(E)\Big).$$

Splitting of functional into three parts, where Per_{1i} and $\mathcal{G}^i_{\alpha,\tau,L}$ depend only on oscillations of the characteristic function of E along the direction e_i ("one-dimensional" terms) and $\mathcal{I}^i_{\alpha,\tau,L}$ is a cross-interaction term given by

$$\frac{2}{d}\int_{[0,L)^d\times\mathbb{R}^d} K_{\alpha,\tau}(\zeta)|\chi_E(x)-\chi_E(x+\zeta_i)||\chi_E(x)-\chi_E(x+\zeta_i^{\perp})|\,\mathrm{d}\zeta\,\mathrm{d}x$$

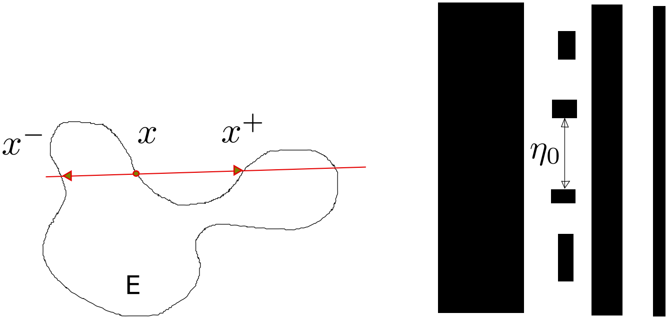
One-dimensional estimates

Let $E \subset \mathbb{R}$ an L-periodic set. Then,

$$-\mathrm{Per}_{1,i}(E,[0,L)) + \mathcal{G}^{1d,i}_{\alpha,\tau,L}(E) \geq$$
$$\geq \sum_{x \in \partial E \cap [0,L)} -1 + C^1_\alpha C^2_\alpha (\min((x^+ - x)^{-(1-\alpha)}, \tau^{-1})$$
$$+ \min((x - x^-)^{-(1-\alpha)}, \tau^{-1})),$$

where

$$x^+ = \inf\{x^+ \in \partial E, \text{ with } x^+ > x\},$$
$$x^- = \sup\{x^- \in \partial E, \text{ with } x^- < x\}$$
$$\text{and } \min_{\alpha<1} C^1_\alpha C^2_\alpha > \bar{C} > 0.$$



(a) (b)

Fix any $\bar{\alpha} < 1$ and any $M > 0$. Then there exists $\eta_0 = \eta_0(M, \bar{\alpha})$ such that whenever

$$\exists x \in \partial E : \quad |x - x^-|, |x - x^+| < \eta_0,$$

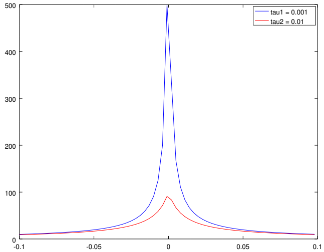
then for any $\alpha \leq \bar{\alpha}$

$$-Per_{1,i}(E) + \mathcal{G}^{1d,i}_{\alpha,\tau,L}(E) > M > 0.$$

Monotonicity of kernel

It is used to show Γ-convergence as $(\alpha, \tau) \downarrow (0, 0)$.

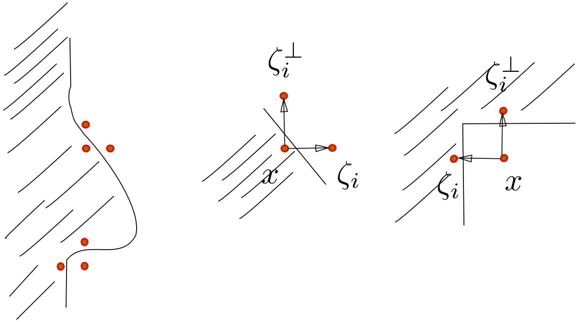
Various $K_{\alpha,\tau}$



Let $\bar{\tau}, \tau, \bar{\alpha}, \alpha \in (0, 1)$. Assume $\bar{\alpha} > \alpha$ and $\bar{\tau} > \tau > 0$. Then $\hat{K}_{\bar{\tau},\alpha}(z) < \hat{K}_{\tau,\alpha}(z)$ and $\hat{K}_{0,\bar{\alpha}}(z) < \hat{K}_{0,\alpha}(z)$.

Stability

Let E be $[0, L)^d$-periodic and L^1-close to stripes in direction e_1. Assume that E is not exactly a union of stripes, and the following happens



Then, for all $\alpha < 1$ (i.e. $p(\alpha) > d + 1$) and τ sufficiently small, the interaction term $\mathcal{I}^1_{\alpha,\tau,L}(E)$ is large, thus making the functional positive and the above configuration not energetically convenient.

Thanks for listening.
Any questions?