CR singular CR images

or

Removable CR singularities

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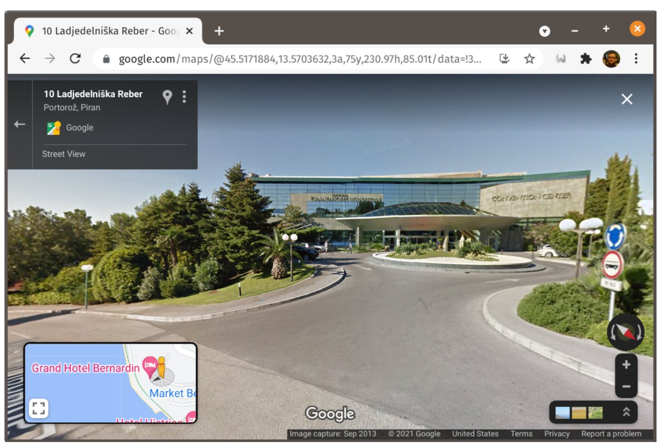

$M \subset \mathbb{C}^m$ real-analytic submanifold.

$$T_p^{0,1} M = \operatorname{span} \left\{ \frac{\partial}{\partial \bar{z}_1} \bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}_m} \bigg|_p \right\} \cap \mathbb{C} \otimes T_p M$$

$\dim T_p^{0,1} M$ constant $\Rightarrow M$ is **CR**.
Else M is **CR singular**.

CR singularities are where $\dim T_p^{0,1} M$ jumps.

CR singular manifolds first studied by Bishop in '65.

Then followed by a long list of papers.

Among others: in $\mathbb{C}^2$ Moser-Webster, Moser, Kenig-Webster, Gong, Huang-Krantz, Huang-Yin, many many others.

In higher dimensions, Dolbeault-Tomassini-Zaitsev, Coffman, Huang, Huang-Yin, Fang-Huang, Gong-Stolovich, Burcea, Slapar, Gong-L., L.-Noell-Ravisankar, etc.

Definition: *Suppose $p \in M$ is a CR singularity and $\exists$ a real-analytic vector bundle $\mathcal{V}$ near p on M such that for every CR point q, $\mathcal{V}_q = T_q^{0,1} M$. Then p is a* ***removable*** *CR singularity.*

If M has only removable CR singularities, then $(M, \mathcal{V})$ is an abstract CR manifold.

We also call such an M a **CR image**:

The inclusion map $\iota: (M, \mathcal{V}) \to \mathbb{C}^m$ is a CR map that is a diffeomorphism onto its image.

As M is real analytic, we can assume that this abstract $(M, \mathcal{V})$ is actually a submanifold of some $\mathbb{C}^n$.

So assume $N \subset \mathbb{C}^n$ is generic (so CR), real-analytic,
F holomorphic $\mathbb{C}^n \to \mathbb{C}^m$, generically of rank n,
$F|_N$ is a diffeomorphism onto $M = F(N)$.

A Bishop surface is a CR image for example:
$F(x, y) = (x + iy, x^2 + y^2 + \lambda(x + iy)^2 + \lambda(x - iy)^2)$
takes $N = \mathbb{R}^2$ to $M : w = |z|^2 + \lambda(z^2 + \bar{z}^2)$.

Every M totally real outside the singularities is a CR image.

When CR dimension is positive, most CR singular manifolds are not CR images.

CR images first noticed in Ebenfelt-Rothschild '07,
studied also in L.-Minor-Shroff-Son-Zhang '14,
and in L.-Noell-Ravisankar '21? ('19 on arxiv).

The extended structure is the CR Nash blowup, see Garrity '00.

$N \subset \mathbb{C}^n$, $F(N) = M \subset \mathbb{C}^m$.

First question is stability:

Can a CR singularity can be perturbed away by keeping the CR structure (the N) fixed and perturbing the embedding F ?

Let $H_p N = T_p N \cap i(T_p N)$.
Let $Z_f = f^{-1}(0)$
Let $C_p Z_f$ be tangent cone of Z_f at p.

Suppose first that $n = m$.

The CR singularities of M are the points $F(p)$ for $p \in N$ where $\det DF|_p = 0$.

Theorem: *If $Z_{\det DF}$ is "transverse" to N in the sense that*

$$H_p N \not\subset C_p Z_{\det DF},$$

then a CR singularity at $F(p)$ cannot be eliminated by perturbing F.

That is, for all G close enough to F (uniform norm), $G(N)$ is CR singular.

Proof is a kind of generalization of Rouchè to the CR setting.

Example: $N = \mathbb{R}^2 \subset \mathbb{C}^2$, $F(x,y) = (x+iy, x^2)$.
$F(N)$ is the parabolic Bishop surface $w = (\operatorname{Re} z)^2$,
$G(x,y) = (x+iy, x^2 + i\epsilon x) \Rightarrow \det DG = -i(2x + i\epsilon)$,
never zero on $N \Rightarrow G(N)$ is CR.
"Transversality" not satisfied: $H_0 N = \{0\}$

Example: "Transversality" is only sufficient:
$N = \mathbb{R}^2 \subset \mathbb{C}^2$, $F(x,y) = (x+iy, x^2+y^2)$.
$F(N)$ is the elliptic Bishop surface $w = |z|^2$, and the CR singularity cannot be removed.
"Transversality" not satisfied: $H_0 N = \{0\}$

Now suppose $n < m$.

As before, $N \subset \mathbb{C}^n$, $F(N) = M \subset \mathbb{C}^m$.

Example: Define M by $w_1 = |z|^6, w_2 = |z|^4$.

M is an image of $N = \mathbb{R}^2$ and $n = 2$, $m = 3$.

M does **not** lie in a 2-dimensional complex submanifold.

Let k be the real codimension of N.

If

$$4n - k < 2(m+1),$$

then near every point of N there exists G arbitrarily close to F such that $G(N)$ is CR.

The inequality is sharp: For any n, k, and m such that $4n - k \geq 2(m+1)$, there exist N and F such that $F(N)$ is CR singular and $G(N)$ is CR singular for all small enough perturbations G of F.

In fact, for such dimensions, such an example exists near any CR image.

Our second question: *Invariants?*

Clearly the biholomorphic invariants of N are invariants of M.

What about invariants of the embedding specifically.

Every codimension 2 CR image $M \subset \mathbb{C}^n$ is biholomorphic to exactly one of the following forms:

1) $w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(\|z\|^3)$,
2) $w = \bar{z}_1 z_2 + O(\|z\|^3)$,
3) $w = |z_1|^2 + a\bar{z}_1^2 + O(\|z\|^3)$, $a \geq 0$,
4) $w = \bar{z}_1^2 + O(\|z\|^3)$,
5) $w = O(\|z\|^3)$.

(Proved in a previous paper also with Noell and Ravisankar)

In fact, for every N, there exists an F such that $F(M)$ is in any one of the above forms.

Example: Consider the quadratic form $\bar{z}_1 z_2$.

Let N be any codimension 2 CR manifold given in $(\zeta, \omega) \in \mathbb{C}^{n-1} \times \mathbb{C}^2$ by

$$\operatorname{Im} \omega_1 = \rho_1(\zeta, \bar{\zeta}, \operatorname{Re} \omega) \qquad \operatorname{Im} \omega_2 = \rho_2(\zeta, \bar{\zeta}, \operatorname{Re} \omega)$$

(ρ_j are $O(2)$)

Define $F(\zeta, \omega)$ by

$$\begin{aligned} z_1 &= \omega_1 + i\omega_2 \\ z_j &= \zeta_{j-1} \qquad (j = 2, \ldots, n-1) \\ w &= \underbrace{(\omega_1 - i\omega_2)}_{\bar{z}_1} \underbrace{\zeta_1}_{z_2} \end{aligned}$$

So the quadratic invariants are independent from the biholomorphic invariants of N.

There exist higher order invariants of M that are not invariants of N.

Example:

$M_1 : w = \bar{z}_1 z_2$
$M_2 : w = \bar{z}_1 z_2 + \bar{z}_1^3$

are not bioholomorphically equivalent, but have the same quadratic form.

Further, both are images of the flat $N = \mathbb{C}^{n-1} \times \mathbb{R}^2$ (given by $\operatorname{Im} \omega_1 = \operatorname{Im} \omega_2 = 0$)

So the two extended CR structures are identical.

The quadratic bits are easy to find given an equation for M:

(just put the quadratic bits in normal form by linear algebra).

The higher order invariants are harder to tease out.

Our last question is identifying removable CR singularities intrinsically.

Consider $\mathbb{C}^3$ only.

Write M as

$w = \rho(z, \bar{z}) = Q(z, \bar{z}) + E(z, \bar{z})$

Q quadratic, $E \in O(3)$.

Proposition: *If* 0 *is a removable CR singularity, then*

$$\overline{T^{0,1}M_{CR}}^{\mathbb{C}\otimes TM}\Big|_0$$

is a complex line in $T_0^{0,1}\mathbb{C}^3$.

That's not an if and only if. It's just a necessary condition (it is "continuity" of the extended structure)

Example: $w = \bar{z}_1 z_2 + \bar{z}_2^3$ satisfies the condition but is not a CR image.

Now suppose that the CR singularity is small, e.g. isolated.

Perhaps counterintuitively, most of the time, if the CR singularity is small, it is not removable.

Proposition: *Suppose* $\bar{\partial} Q \not\equiv 0$, M *does not correspond to a parabolic Bishop surface, and the set of CR singularities of dimension less than 2. Then* 0 *is not a removable CR singularity of* M.

We say that M corresponds to a parabolic Bishop surface if we can put M in a form where $Q(z, \bar{z}) = \frac{1}{2}(z_1 + \bar{z}_1)^2$.

Example:

$$w = z_1\bar{z}_1 + \frac{\bar{z}_1^2}{2} + \frac{i}{2}\left(z_1\bar{z}_1^2 + 2\bar{z}_1 z_2 \bar{z}_2\right)$$

has an isolated CR singularity, corresponds to a parabolic Bishop surface, and the singularity **is not** removable.

Example:

$$w = \frac{1}{2}(z_1 + \bar{z}_1)^2 - z_1^2 z_2^2 \bar{z}_1^2 \bar{z}_2^2 - i z_1^2 z_2^4 \bar{z}_1 \bar{z}_2^4 + \frac{i}{3} z_1^2 \bar{z}_1^3$$
$$+ i z_2^2 \bar{z}_1 \bar{z}_2^2 + i z_1 z_2^2 \bar{z}_2^2 + \frac{1}{3} z_1^2 z_2^6 \bar{z}_2^6 - \frac{1}{2} z_2^4 \bar{z}_2^4$$

has an isolated CR singularity, corresponds to a parabolic Bishop surface, and the singularity **is** removable. (That's the simplest example we could find)

In general, the trick is to extend the CR vector field:

Proposition: *The origin is a removable CR singularity if and only if either*

$$\frac{\rho_{\bar{z}_1}}{\rho_{\bar{z}_2}} \quad \text{or} \quad \frac{\rho_{\bar{z}_2}}{\rho_{\bar{z}_1}}$$

is (extends to) a real-analytic function near the origin.

Extending as C^ℓ for any finite ℓ is not sufficient, although extending as C^∞ implies C^ω by Malgrange, so smooth extension is also sufficient.

Thank you for your attention!

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