Automorphisms of direct products of some circulant graphs

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Abstract. The direct product of two graphs X and Y is denoted $X \times Y$. This is a natural construction, so any isomorphism from X to X' can be combined with any isomorphism from Y to Y' to obtain an isomorphism from $X \times Y$ to $X' \times Y'$. Therefore, the automorphism group $\operatorname{Aut}(X \times Y)$ contains a copy of $(\operatorname{Aut} X) \times (\operatorname{Aut} Y)$. It is not known when this inclusion is an equality, even for the special case where $Y = K_2$ is a connected graph with only 2 vertices.

Recent work of B. Fernandez and A. Hujdurović solves this problem when X is a “circulant” graph with an odd number of vertices (and $Y = K_2$). We will present a short, elementary proof of this theorem.

Assumption

*X and Y are connected, **circulant graphs**, with more than 1 vertex,*
*(or Cayley graphs on **abelian** groups)*

Definition of *direct product* $X \times Y$:

- vertices are ordered pairs $(x, y) \in V(X) \times V(Y)$
- $(x_1, y_1) — (x_2, y_2) \iff x_1 — x_2$ and $y_1 — y_2$

(categorical, tensor, Kronecker, ...)

Exercise

1. $\operatorname{Aut} X \times \operatorname{Aut} Y \subseteq \operatorname{Aut}(X \times Y)$: $\varphi(x, y) = (\alpha(x), \beta(y))$.
2. $\operatorname{Aut}(X \times X) \neq \operatorname{Aut} X \times \operatorname{Aut} X$. $\varphi(x_1, x_2) = (x_2, x_1)$.

Assumption

$|V(X)|$ is relatively prime to $|V(Y)|$. ($|V(X)|$ *is odd*)

X and Y are connected, circulant graphs, with more than one vertex.
$|V(X)|$ is relatively prime to $|V(Y)|$

Theorem (Sabidussi-Vizing 1960/1963, Dörfler-Imrich 1970)

$\mathrm{Aut}(X \square Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$. $\mathrm{Aut}(X \boxtimes Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ *iff* *technical condition*

Theorem (Fernandez-Hujdurović 2020$^+$, Morris 2021)

$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ *iff X and Y are twin-free:* *no vertices have same neighbours*

Previous (Dörfler 1974): X and Y **not** bipartite (e.g., both odd order).

Main case of theorem (implies the general case) is when $Y = K_2$:

Definition (Marušič-Scapellato-Salvi 1989)

X is ***stable*** if $\mathrm{Aut}(X \times K_2) = \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$.
(canonical bipartite double cover)

Proof (some trivial observations)

Definition (graph Y^2_{odd} on same vertices as Y)

$y_1 \overset{\text{odd } 2}{\text{—}} y_2$ if $\exists$ **odd** # of paths of length 2 from y_1 to y_2.

Obs. $\operatorname{Aut} Y \subseteq \operatorname{Aut} Y^2_{\text{odd}}$.

Assump. $Y = \operatorname{Cay}(G; S)$ with G abelian.

Obs. path of length 2: $y_1 \text{—} y_1 + s \text{—} y_1 + s + t = y_2$

$\implies y_1 \text{—} y_1 + t \text{—} y_1 + s + t = y_2$ also path of length 2.

$\therefore$ paths of length 2 from y_1 to y_2 come in pairs

unless $s = t$, i.e., $y_1 \text{—} y_2$ in $\operatorname{Cay}(G; 2S)$ where $2S = \{\, 2s \mid s \in S \,\}$.

Lem [known?]. Assume $2s \neq 2t$ for all $s, t \in S$, such that $s \neq t$.
Then $Y^2_{\text{odd}} = \operatorname{Cay}(G; 2S)$, so $\operatorname{Aut} \operatorname{Cay}(G; S) \subseteq \operatorname{Aut} \operatorname{Cay}(G; 2S)$.

Can replace 2 with any $k \in \mathbb{Z}^+$, but proof is a bit more complicated.

Theorem (Fernandez-Hujdurović 2020$^+$, Morris 2021)

*$X = \mathrm{Cay}(G;S)$ with G abelian of **odd order*** *(connected, twin-free)*
$\Rightarrow \quad \mathrm{Aut}(X \times K_2) = \mathrm{Aut}\,X \times \mathrm{Aut}\,K_2.$

Lemma (easy)

If G' abelian, and $\forall s', t' \in S'$: $2s' = 2t' \implies s' = t'$, then
$\mathrm{Aut}\,\mathrm{Cay}(G';S') \subseteq \mathrm{Aut}\,\mathrm{Cay}(G';\mathbf{2}S')$ *where* $2S' = \{\, 2s' \mid s' \in S' \,\}$.

Proof of Theorem. $X \times K_2 = \mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{1\}\big)$. (can take as definition)
$2(s,1) = 2(t,1) \Rightarrow (2s,0) = (2t,0) \Rightarrow 2s = 2t \Rightarrow s = t.$

$$\begin{aligned}
&\mathrm{Aut}\ \mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{1\}\big)\\
&\quad \subseteq \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ \mathbf{2}(S \times \{1\})\big)\\
&\quad = \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ 2S \times \{0\}\big)\\
&\quad \subseteq \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ \mathbf{2^k}S \times \{0\}\big)\\
&\quad = \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{0\}\big) \qquad (\text{choose } 2^k \equiv 1 \pmod{|G|})
\end{aligned}$$

So restriction to bottom layer is in $\mathrm{Aut}\,X$: $\varphi(x,0) = \big(\alpha(x), 0\big)$.
Since there are no twins: $\varphi(x,1) = \big(\alpha(x), 1\big)$. (Exercise) □

Theorem (Fernandez-Hujdurović 2020^{+}, Morris 2021)

$\mathrm{Aut}(X \times K_2) = \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$ *iff X is twin-free:* *no vertices have same neighbours*

Assumption

X is connected, ***circulant graph*** *(or Cayley graph on* ***abelian*** *group)*
and **$|V(X)|$ is odd.**

Remark (Hujdurović-Mitrović 2020 (unpublished – see [Morris 2021]))

Cannot delete "abelian." (Counterexample with 21 vertices.)

Open problem (Wilson 2008)

Which circulant graphs are stable?

Assume X is connected, circulant graph with no twin vertices.
Also assume $|V(X)|$ is even, but X is **not** bipartite.
Characterize cases where $\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$. (uncommon)

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Aut($X \times K_2$) when X is a circulant graph

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