A Step towards Non-Presentable Models of Homotopy Type Theory

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Types and Categories

Two foundations for mathematics

1. **Types:** Basic building block types with various type constructors (Π-types, Σ-types, ...)
2. **Categories:** Categories with various objects with universal properties (limits, Cartesian closure, ...)

We can relate them:

- λ-**calculus** and **Cartesian closed categories**.
- **Higher order type theories** and **elementary toposes**.
- Examples: Grothendieck toposes, filter quotient toposes, realizability toposes

Lambek, *Cartesian Closed Categories and Typed Lambda-calculi*. Combinators and Func. Prog. Lang. (1985)

Lambek, Scott, *Introduction to higher order categorical logic* (1988)

Homotopy Types and Higher Categories

Expect two foundations for higher mathematics

1. **Homotopy Types:** type theory with additional conditions (univalence, ...)
2. **Higher Categories:** Categories up to homotopy with weak universal properties (homotopy limits, ...)

We *expect* to relate them:

- Spaces model homotopy type theory (Lumsdaine-Kapulkin)
- Grothendieck higher toposes model homotopy type theory (Shulman)
- Intensional type theories with Σ-types correspond to finitely complete quasi-categories (Kapulkin-Szumilo)
- Intensional type theories with Π-types give us locally Cartesian closed quasi-categories (Kapulkin)
- ...

Filter quotients as Models

- **Goal:** Construct new models of homotopy type theory.
- **Motivation:** (Adelman-Johnstone) Given an elementary topos $\mathcal{E}$ and a filter of subobjects $\mathcal{F}$ we can construct a filter quotient elementary topos $\mathcal{E}_{\mathcal{F}}$ which is a new model of higher order type theory.
- **Expectation:** Higher categorical filter quotients give us models of homotopy type theory.
- **What we know:** (R) For a given higher topos $\mathcal{E}$ and filter $\mathcal{F}$ we can construct filter quotient higher toposes $\mathcal{E}_{\mathcal{F}}$.
- **Next Step:** Show these filter quotient higher toposes are models of homotopy type theory.

Enter Model Categories

1. Type theory is inherently strict
2. Higher category theory is up to homotopy

Solution: **model categories** or **fibration categories** used as a bridge. They are strict categories with additional data that help us recover higher categories.

- Model categories are used to prove Grothendieck higher toposes model homotopy type theories
- Fibrations categories are used to relate extensional type theories with quasi-categories

New Model Structures

This leads to following concrete question:

Question

Let $\mathcal{M}$ be a model category and $\mathcal{F}$ a filter on $\mathcal{M}$. Then does there exist a model structure on the filter quotient $\mathcal{M}_{\mathcal{F}}$ that inherits properties from $\mathcal{M}$?

Let us figure this out!

Filters

Definition

Let $\mathcal{C}$ be a finitely complete category with terminal object 1. A **filter of subobjects** $\mathcal{F}$ is a sub-poset of the poset of subobjects of 1 with the following properties.

1. **Maximum:** $1 \in \mathcal{F}$
2. **Downward Directed:** If $U, V \in \mathcal{F}$ then there exists $W \in \mathcal{F}$, with $W \leq U, V$.
3. **Upwards Closed:** If $U \leq V$ and $U \in \mathcal{F}$ Then $V \in \mathcal{F}$.

Example

If $\mathcal{C} = \mathcal{S}\mathrm{et}^{\mathbb{N}}$, then $(*, *, ...)$ is terminal and we can take $\mathcal{F}$ the cofinite subsets.

Filter Quotient Categories

Theorem

Let $\mathcal{C}$ be a finitely complete category with filter $\mathcal{F}$. Then there exists a filter quotient category $\mathcal{C}_{\mathcal{F}}$ which has the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{F}}}(A, B) = \mathrm{colim}_{U \in \mathcal{F}} \mathrm{Hom}_{\mathcal{C}}(U \times A, U \times B).$$

Example

With $\mathcal{C} = \mathcal{S}\mathrm{et}^{\mathbb{N}}$, and $\mathcal{F}$ as before, $\mathcal{C}_{\mathcal{F}}$ has the morphisms of $\mathcal{C}$ up to "eventual equality".

Denseness and Model Categories

A subset $\mathcal{G} \subseteq \mathcal{F}$ is **dense** if for any $U \in \mathcal{F}$, there exists $V \in \mathcal{G}$ such that $V \leq U$.

Theorem (K-R)

Let $\mathcal{M}$ be a model category. Suppose that the following sets of elements $U \in \mathcal{F}$ are dense:

1. $- \times U$ *preserves finite colimits*
2. $- \times U$ *preserves weak equivalences and cofibrations*
3. $- \times U$ *preserves weak equivalences and fibrations*

Then $\mathcal{M}_F$ is a model category with the induced fibrations, cofibrations and weak equivalences.

Property preservation

Theorem (K-R)

Let $\mathcal{M}$ be a model category with a filter $\mathcal{F}$ satisfying the appropriate conditions.

- *If $\mathcal{M}$ is (left/right) proper then $\mathcal{M}_{\mathcal{F}}$ is (left/right) proper.*
- *If $\mathcal{M}$ is simplicial and $\mathcal{F}$ is compatible with simpliciality, then $\mathcal{M}_{\mathcal{F}}$ is simplicial.*
- *If $\mathcal{M}$ satisfies finite descent, then $\mathcal{M}_{\mathcal{F}}$ also satisfies finite descent.*

Filter Products

Let I be a set and $\mathcal{F}$ a filter on the power set of I. Then $\mathcal{F}$ is a filter of subobjects on the product Kan model category $\prod_I \mathrm{s}\mathcal{S}\mathrm{et}$ satisfying the density conditions and simpliciality conditions. So, we have following corollary

Corollary

There exists a proper, simplicial model structure on the filter product of simplicial sets $\prod_{\mathcal{F}} \mathrm{s}\mathcal{S}\mathrm{et}$ induced by the Kan model structure.

Filter of Functors

- Everything said up until here is a special case of a more general approach to filter quotients via **filter of functors**.
- This allows us to generalize the study of filter quotients to pointed categories that arise in algebra and homotopy theory.

Why we did it

- We want to understand the relation between type theories and higher categories.
- In particular, we want to understand the relation between filter quotient higher categories and homotopy type theories
- We study this relation via model categories, which are a strictification of higher categories.
- This justifies carefully studying filter quotient model categories.

What we did

- We study conditions on filters in model categories that induce model structures on filter quotients.
- We apply this in particular to filter products.
- We prove that under suitable conditions the filter quotient construction preserves model categorical properties.

What we want to do

- Look at further topos theoretic properties of the filter quotient construction, in particular related to work of Shulman.
- Use the more general filter of functor quotients to further study filter quotients of pointed and triangulated categories.
- Use the filter quotient to construct new model categories that arise in stable homotopy theory or homological algebra.

Thank you! Questions?

Thank You!

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