Asymptotic analysis of rigidity constraints modeling fiber-reinforced composites

Antonella Ritorto

KU Eichstätt-Ingolstadt

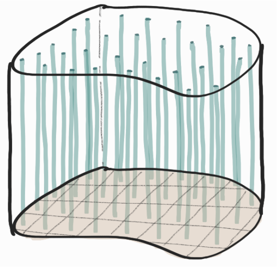
antonella.ritorto@ku.de

Joint work with
Dominik Engl (Utrecht University) and
Carolin Kreisbeck (KU Eichstätt-Ingolstadt)

Energy methods and their applications in material science
June 23, 2021

Motivation

Type of composites: Solids reinforced by vanishing thin long rigid fibers.



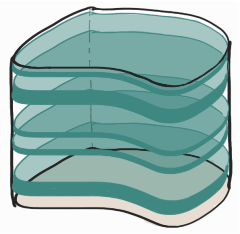
Main features:

- special geometry,
- high-contrast: rigid vs. soft.

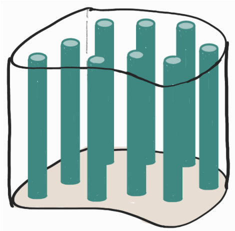
Goal: Study mathematical rigorously their macroscopic behavior.

Impact: Reinforced composites play an important role in various industrial applications such as lightweight-composites production.

Some related works



- **Christowiak & Kreisbeck'17, '20**
  Layered materials



- **El Jarroudi'13**
  Second gradient nonlinear elastic material.
- **Pideri & Seppecher'97**
  Second gradient linear elastic material

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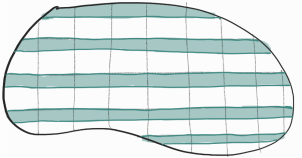

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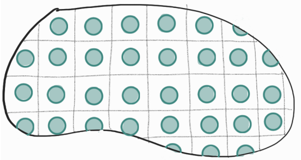

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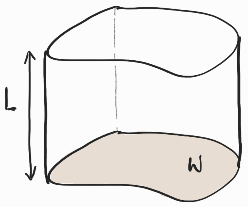
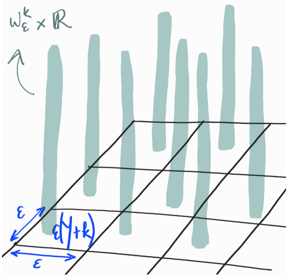
Set-up

Geometry of the body



- $\Omega = \omega \times (0, L)$ a material body with height $L > 0$
- $\omega \subset \mathbb{R}^2$ a bounded Lipschitz domain
- $x = (x', x_3)$ with $x' \in \omega$, $x_3 \in (0, L)$

Distribution of the fibers



- $Y = (0, 1]^2$ the unit cell
- For $k \in \mathbb{Z}^2, \varepsilon > 0$:
  - ω_ε^k the fiber cross-section
  - $\omega_\varepsilon^k \subset \varepsilon(Y + k)$
- $Y_\varepsilon^{\text{rig}} = \bigcup_{k \in \mathbb{Z}^2} \omega_\varepsilon^k \times \mathbb{R}$ all the rigid fibers

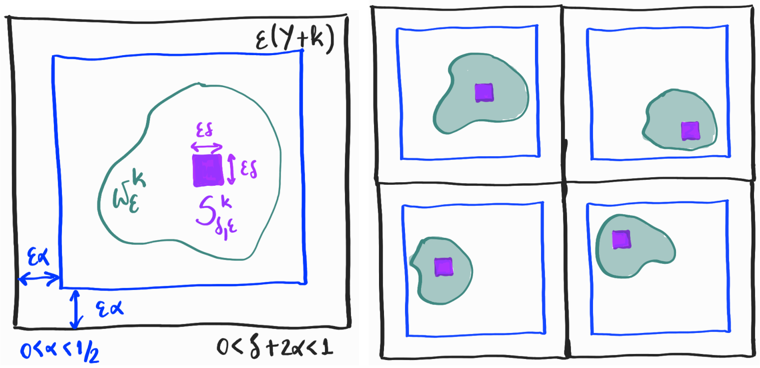
Technical assumptions

Cross section of the body:

- $\exists\, \phi\colon B_1(0) \to \omega$ bi-Lipschitz

Cross section of the fibers:

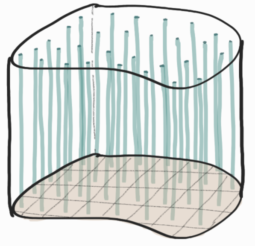
- $\omega_\varepsilon^k \subset \varepsilon([\alpha, 1-\alpha)^2 + k) \subset \varepsilon(Y + k)$
- $\omega_\varepsilon^k \supset S_{\delta,\varepsilon}^k$ of side length $\varepsilon\delta$

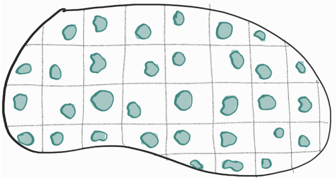


Goal: Characterize the macroscopically attainable deformations for these structures.

Mathematically speaking:

Determine the $W^{1,p}(\Omega;\mathbb{R}^3)$-weak limits u of sequences $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^3)$ s.t. $\nabla u_\varepsilon \in \mathrm{SO}(3)$ in every fiber $\omega_\varepsilon^k \times (0, L) \subset \Omega$.





One may think of $(u_\varepsilon)_\varepsilon$ as a sequence of **uniformly bounded energy** for functionals of the form

$$u \mapsto \int_{\Omega \setminus Y_\varepsilon^{\mathrm{rig}}} W_{\mathrm{soft}}(\nabla u)\,\mathrm{d}x + \int_{Y_\varepsilon^{\mathrm{rig}} \cap \Omega} W_{\mathrm{rig}}(\nabla u)\,\mathrm{d}x,$$

- $W_{\mathrm{soft}} : \mathbb{R}^{3\times 3} \to [0, \infty)$ the elastic energy density
  - frame-indifference
  - suitable growth and coercivity
  - vanishing on the identity
- $W_{\mathrm{rig}} : \mathbb{R}^{3\times 3} \to [0, \infty]$ is given by

$$W_{\mathrm{rig}}(F) = \begin{cases} 0 & \text{for } F \in \mathrm{SO}(3), \\ \infty & \text{otherwise.} \end{cases}$$

Main result

Denote the set of such limits as $\mathcal{A}_0$, i.e.

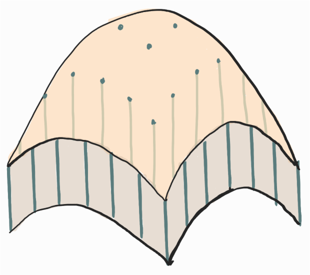
$$\mathcal{A}_0 = \{u \in W^{1,p}(\Omega;\mathbb{R}^3): \text{ there exists } (u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^3), \\ u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^3), \\ \nabla u_\varepsilon \in \mathrm{SO}(3) \text{ a.e. in } Y_\varepsilon^{\mathrm{rig}} \cap \Omega\}$$

Theorem (Engl-Kreisbeck-R., 2021)

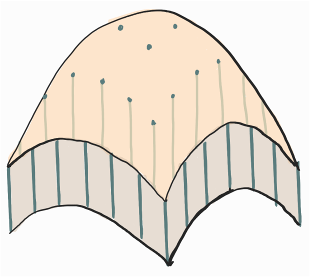
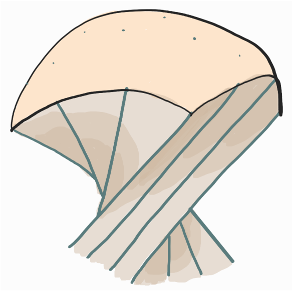
Let $p > 2$. Under the previous assumptions,

$$\mathcal{A}_0 = \{u \in W^{1,p}(\Omega;\mathbb{R}^3) : \exists\, \Sigma \in W^{1,p}(\omega;\mathcal{S}^2),\, d \in W^{1,p}(\omega;\mathbb{R}^3), \\ \text{s.t. } u(x) = x_3\Sigma(x') + d(x') \\ \text{for a.e. } x \in \Omega\}.$$

Examples

Incompressible deformation	
$$u(x) = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ -(x_1^2 + x_2^2) \end{pmatrix}$$ 	

Examples

Incompressible deformation	Non-incompressible deformation
$u(x) = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ -(x_1^2 + x_2^2) \end{pmatrix}$	$u(x) = \frac{x_3}{\sqrt{2(x_1^2+x_2^2+1)}} \begin{pmatrix} -x_1 - x_2 \\ x_1 - x_2 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$
	

Part 1: Necessity

Let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be s.t. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and

$$\nabla u_\varepsilon \in \mathrm{SO}(3) \text{ a.e. in } Y_\varepsilon^{\mathrm{rig}} \cap \Omega. \qquad (\star)$$

Then,

$$u(x) = x_3\Sigma(x') + d(x') \qquad \text{a.e. } x \in \Omega$$

with

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Remark: Same statement holds if $(\star)$ is replaced by

$$\int_{Y_\varepsilon^{\mathrm{rig}}\cap\Omega} \mathrm{dist}^p(\nabla u_\varepsilon, \mathrm{SO}(3))\,\mathrm{d}x \leq C\varepsilon^\beta$$

for $C > 0$ and $\beta > 2p$.

Part 1. (Some) ideas of the proof

• **Classical rigidity** [Reshetnyak'67]: $\exists\, R^k_\varepsilon \in \mathrm{SO}(3)$ and $b^k_\varepsilon \in \mathbb{R}^3$ s.t.

$$u_\varepsilon(x) = R^k_\varepsilon x + b^k_\varepsilon \quad \text{in } \omega^k_\varepsilon \times \mathbb{R}$$

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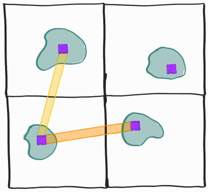
- **Auxiliary sequence**:

$$\Sigma_\varepsilon(x') = \sum_k R_\varepsilon^k e_3 \mathbb{1}_{\varepsilon(Y+k)}(x')$$

  - Compactness: $\Sigma_\varepsilon \to \Sigma$ in $L^p(\omega;\mathbb{R}^3)$
    - Fréchet-Kolmogorov Theorem
    - Estimates of neighboring squares

$$\boxed{\|\Sigma_\varepsilon(\cdot+\xi) - \Sigma_\varepsilon\|^p_{L^p(\omega;\mathbb{R}^3)} \le C(|\xi|^p + \varepsilon^p)}$$



  - Regularity: $\Sigma \in W^{1,p}(\omega;\mathcal{S}^2)$

cf. [Friesecke-James-Müller'02], [Christowiak-Kreisbeck'17]

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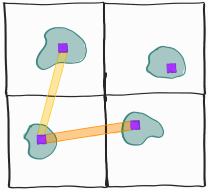

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cf. [Friesecke-James-Müller'02], [Christowiak-Kreisbeck'17]

- **Comparison with the auxiliary sequence**:
  - $\Sigma = \partial_3 u \quad \Rightarrow u(x) = x_3\Sigma(x') + d(x') \quad$ ■

Part 2: Sufficiency

Let $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ be s.t.

$$u(x) = x_3\Sigma(x') + d(x')$$

with

- $\Sigma \in W^{1,p}(\omega;\mathcal{S}^2)$
- $d \in W^{1,p}(\omega;\mathbb{R}^3)$

Then, $\exists\ (u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega;\mathbb{R}^3)$ s.t. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and

$$\nabla u_\varepsilon \in \mathrm{SO}(3) \text{ a.e. in } Y_\varepsilon^{\mathrm{rig}} \cap \Omega.$$

Part 2. (Some) ideas of the proof

- **Lifting argument**:

$$u(x) = x_3\Sigma(x') + d(x') \qquad \Leftrightarrow \qquad u(x) = R(x')x + b(x')$$

with $R \in W^{1,p}(\omega; \mathrm{SO}(3))$

and $b \in W^{1,p}(\omega; \mathbb{R}^3)$.

see [Bethuel-Chiron'07]

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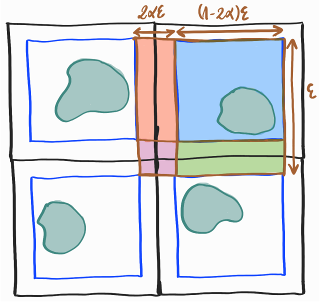
- **Approximation of the identity**: $\exists\ (\varphi_\varepsilon)_\varepsilon \in W^{1,\infty}(\omega; \mathbb{R}^2)$ s.t.
  - $\sup_\varepsilon \|\nabla'\varphi_\varepsilon\|_{L^\infty(\omega;\mathbb{R}^{2\times 2})} < \frac{1}{\alpha}$
  - $\varphi_\varepsilon \to \mathrm{id}_{\mathbb{R}^2}$ uniformly
  - φ_ε is constant a.e. in $(Y_\varepsilon^{\mathrm{rig}})' \cap \omega$.

$$\nabla'\varphi_\varepsilon(x) = \begin{cases} \frac{1}{2\alpha}(e_1|e_2) \\ \frac{1}{2\alpha}(e_1|0) \\ 0 \\ \frac{1}{2\alpha}(0|e_2) \end{cases}$$



Part 2. (Some) ideas of the proof

- **Lipschitz case**: R and b are Lipschitz

$$\boxed{u_\varepsilon(x) = R(\varphi_\varepsilon(x'))x + b(\varphi_\varepsilon(x'))}$$

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- **Non-Lipschitz case**:
  - Density argument: R_η and b_η are Lipschitz

$$u_\eta(x) = R_\eta(x')x + b_\eta(x'), \qquad u_\eta \to u \text{ in } W^{1,p}(\Omega;\mathbb{R}^3)$$

Candidate: $u_{\eta,\varepsilon}(x) = R_\eta(\varphi_\varepsilon(x'))x + b_\eta(\varphi_\varepsilon(x'))$

Problem: proving that $(u_{\eta,\varepsilon})_\varepsilon$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$

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  - Shifting the approximation of the identity: $\exists\, a_\varepsilon \in B_\varepsilon(0) \subset \mathbb{R}^2$ s.t.

$$\boxed{u^{a_\varepsilon}_{\eta,\varepsilon}(x) = R_\eta(\varphi_\varepsilon(x') + a_\varepsilon)x + b_\eta(\varphi_\varepsilon(x') + a_\varepsilon)}$$

verifying $\|\nabla u^{a_\varepsilon}_{\eta,\varepsilon}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq C$, cf. [Conti-Dolzmman'15] ■

Macroscopic rigid body motion via partial regularization

Theorem (Engl-Kreisbeck-R., 2021)

Let $p > 1$ *and* $u \in W^{1,p}(\Omega; \mathbb{R}^3)$. *Suppose that there exists* $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ *such that* $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega; \mathbb{R}^3)$, $\nabla u_\varepsilon \in \mathrm{SO}(3)$ *a.e. in* $Y_\varepsilon^{\mathrm{rig}} \cap \Omega$, *and*

$$\sup_{\varepsilon > 0} \max_{i,j \in \{1,2\}} \|\partial_i \partial_j u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^3)} < \infty.$$

Then, u *is a rigid body motion, i.e.,*

$$u(x) = Rx + b, \qquad \text{with } R \in \mathrm{SO}(3),\ b \in \mathbb{R}^3.$$

Idea of the proof: similar to Part 1 of the main result

Without partial reg.: $\|\Sigma_\varepsilon(\cdot + \xi) - \Sigma_\varepsilon\|^p_{L^p(\omega;\mathbb{R}^3)} \le C(|\xi|^p + \varepsilon^p)$

With partial reg.: $\|\Sigma_\varepsilon(\cdot + \xi) - \Sigma_\varepsilon\|^p_{L^p(\omega;\mathbb{R}^3)} \le C(|\xi|^p \varepsilon^p + \varepsilon^{2p})$ ■

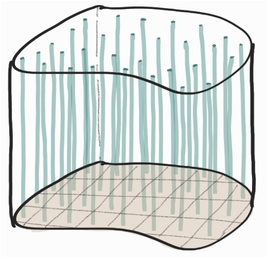
Wrapping up

• Without any additional regularity, the macroscopically attainable deformations for our model of fiber-reinforced materials are characterized by locally preserving length in the fiber direction.

• With partial regularization in the cross-section variables, only rigid body motions can occur.

Outlook: Basis for Γ-convergence results paving the way for homogenization for fiber-reinforced materials.

Thank you for your attention!



Reference: Asymptotic analysis of deformation behavior in high-contrast fiber-reinforced materials: rigidity and anisotropy (with D. Engl & C. Kreisbeck). *Preprint arXiv:2105.03971*