# 1-skeleton of the polytope of pyramidal tours with step-backs

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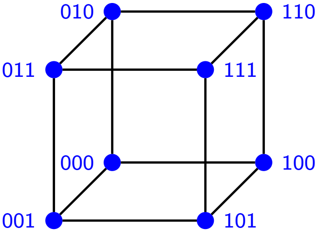
# 1-skeleton of a polytope




### Definition

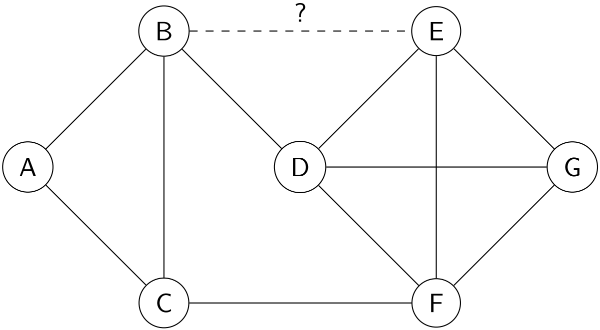
The 1-*skeleton* of a polytope $P$ is the graph whose vertex set is the vertex set of $P$ and edge set is the set of 1-faces of $P$.

### Comment

We consider 0/1-polytopes that are associated with combinatorial optimization problems and arise from LP formulations of problems.

# Objects of interest (I)

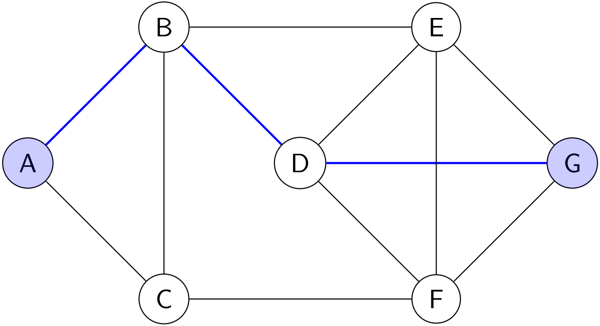
1. Adjacency relation




- It is quite useful for research of 1-skeletons.
- Serves as a neighborhood structure and the basis for edge-following algorithms.

# Objects of interest (II)

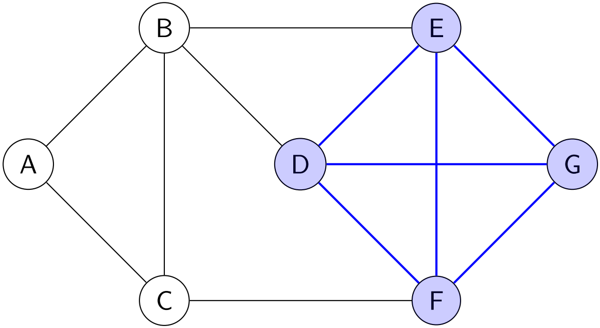
2. Graph diameter (the maximum edge distance between any pair of vertices)




The diameter serves as the lower bound on the number of steps of the edge-following algorithms like the simplex-method.

# Objects of interest (III)

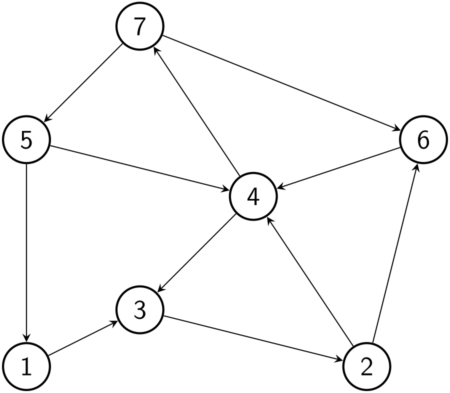
3. Clique number (the number of vertices in the largest clique)




The clique number serves as the lower bound on the number of steps in a special class of direct-type algorithms (some of the branch and bound algorithms, dynamic programming, etc.).

# Traveling salesperson problem (TSP)

ASYMMETRIC TRAVELING SALESPERSON PROBLEM. Given a complete weighted digraph $D_n = (V, E)$, it is required to find a Hamiltonian tour of minimum weight.



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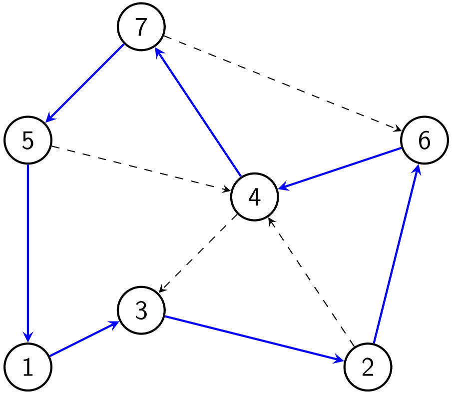
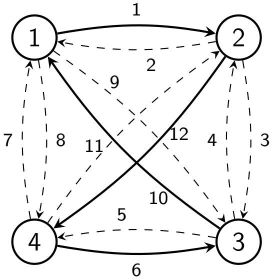


# Traveling salesperson polytope

We consider a complete graph $D_n$ with the edge set $E$. Let $HT_n$ be the set of all Hamiltonian tours in $D_n$. With each tour $x \in HT_n$ we associate a characteristic vector $x^v \in \mathbb{R}^E$ by the following rule:

$$x_e^v = \begin{cases} 1, & \text{if an edge } e \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$




$$x^v = (1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1)$$

The polytope

$$\text{ATSP}(n) = \text{conv}\{x^v \mid x \in HT_n\}$$

is called *the asymmetric traveling salesperson polytope*.

*The symmetric traveling salesperson polytope* $\text{TSP}(n)$ is defined similarly.

# What is known?

Adjacency (Papadimitriou, 1978)

The question of whether two vertices of the polytopes ATSP($n$) or TSP($n$) are nonadjacent is NP-complete.

Diameter (Padberg, Rao, 1974; Rispoli, Cosares, 1998)

The diameter of ATSP($n$) 1-skeleton equals 2 for all $n \geq 6$.

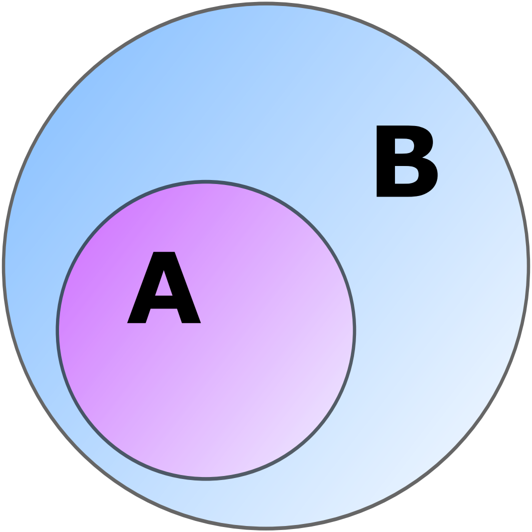
The diameter of TSP($n$) 1-skeleton is bounded above by 4.

Clique number (Bondarenko, 1983)

The clique number of the ATSP($n$) 1-skeleton is superpolynomial in dimension:

$$\omega(\mathrm{ATSP}(n)) \geq 2^{\left(\sqrt{n}-9\right)/2}.$$

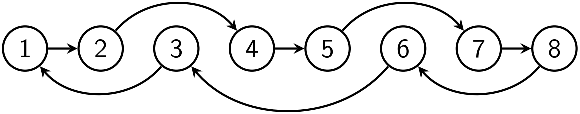
# Special cases



# Pyramidal tours

## Definition

A Hamiltonian tour is called *a pyramidal* if the salesperson starts in the city 1, then visits some cities in ascending order, reaches the city $n$, and returns to the city 1 visiting the remaining cities in descending order.




## Nice properties (Aizenshtat, Kravchuk, 1968; Klyaus, 1976; Gilmore, Lawler, and Shmoys, 1985)

1. A minimum cost pyramidal tour can be determined in $O(n^2)$ time by dynamic programming.
2. There exists certain combinatorial structures of distance matrices that guarantee the existence of the shortest tour that is pyramidal.

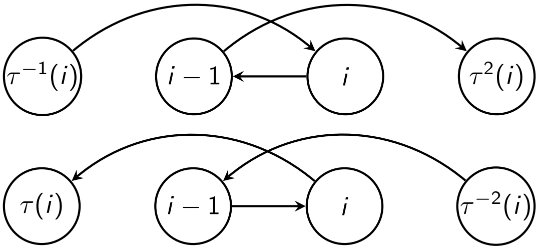
# Pyramidal tours with step-backs

**Definition (Enomoto, Oda, Ota, 1998)**

Let $\tau$ be a Hamiltonian tour. A city $i$ satisfying $\tau^{-1}(i) < i$ and $\tau(i) > i$ is called *a peak*. *A step-back peak* is a city $i$ such that

$$\tau^{-1}(i) < i,\ \tau(i) = i-1 \text{ and } \tau^2(i) > i,\ \text{ or } \tau^{-2}(i) > i,\ \tau^{-1}(i) = i-1 \text{ and } \tau(i) < i.$$

*A proper peak* is a peak $i$ which is not a step-back peak. *A pyramidal tour with step-backs* is a Hamiltonian tour $\tau$ which has exactly one proper peak $n$.



# Pyramidal polytopes

We denote by $PT_n$ the set of all pyramidal tours and by $PSBT_n$ the set of all pyramidal tours with step-backs in the complete digraph $D_n = (V, E)$. With each pyramidal tour (with step-backs) $x \in PT_n$ ($x \in PSBT_n$) we associate a characteristic vector $x^v \in \mathbb{R}^E$ by the following rule:

$$x_e^v = \begin{cases} 1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$

The polytope

$$\text{PYR}(n) = \text{conv}\{x^v \mid x \in PT_n\}$$

is called the *polytope of pyramidal tours*.
The polytope

$$\text{PSB}(n) = \text{conv}\{x^v \mid x \in PSBT_n\}$$

is called the *polytope of pyramidal tours with step-backs*.

# 1. Vertex adjacency – auxiliary statements

Let $x$ and $y$ be two pyramidal tours with step-backs. We denote by $x^v$ and $y^v$ the corresponding vertices of the $\mathrm{PSB}(n)$ polytope and by $x \cup y$ a regular directed multigraph that contains all edges of both tours $x$ and $y$.

### Sufficient condition for nonadjacency

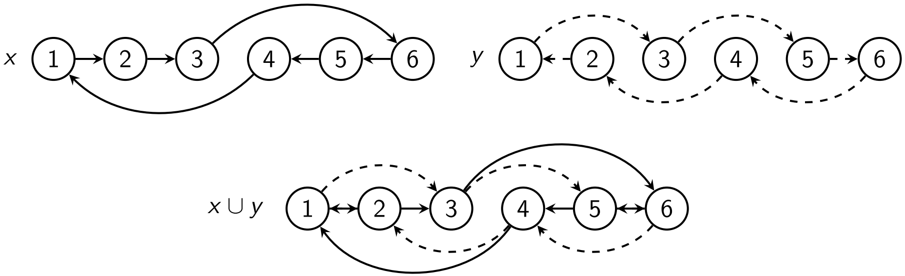
Given two tours $x$ and $y$, if the multigraph $x \cup y$ includes a pair of edge-disjoint pyramidal tours with step-backs, different from $x$ and $y$, then the corresponding vertices $x^v$ and $y^v$ of the polytope $\mathrm{PSB}(n)$ are not adjacent.

### Necessary condition for nonadjacency

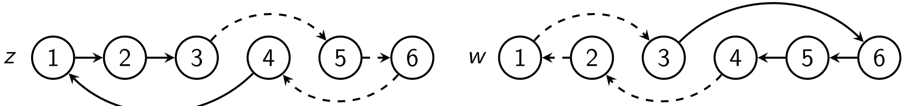
If the vertices $x^v$ and $y^v$ of the polytope $\mathrm{PSB}(n)$ are not adjacent, then the multigraph $x \cup y$ includes at least two pyramidal tours with step-backs, different from $x$ and $y$.

# 1. Vertex adjacency – a combinatorial problem

**Input:**




**Output:**

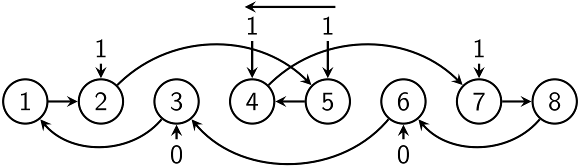



The corresponding vertices $x^v$ and $y^v$ of the polytope $\mathrm{PSB}(n)$ are not adjacent.

# Pyramidal encoding

With every pyramidal tour with step-backs $x$ we associate a vector $x^{0,1,sb}$ of length $n-2$ by the following rule:

$$x_i^{0,1,sb} = \begin{cases} 1, & \text{if } i \text{ is visited by } x \text{ in ascending order,} \\ \overleftarrow{1\ 1}, & \text{if } i \text{ is a step-back peak in ascending order,} \\ 0, & \text{if } i \text{ is visited by } x \text{ in descending order,} \\ \overrightarrow{0\ 0}, & \text{if } i \text{ is a step-back peak in descending order.} \end{cases}$$




$$\Updownarrow$$

$$x^{0,1,sb} = \left\langle 1, 0, \overleftarrow{1, 1}, 0, 1 \right\rangle$$

# 1. Vertex adjacency – final criterion

We consider 12 blocks of the following form (a wavy line means that the corresponding coordinate can either contain a step-back or not):

$$U_{11} = \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle,\ U_{00} = \left\langle \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle,\ U_{1111} = \left\langle \begin{matrix} \overleftarrow{1\ \ 1} \\ \overleftarrow{1\ \ 1} \end{matrix} \right\rangle,\ U_{0000} = \left\langle \begin{matrix} \overrightarrow{0\ \ 0} \\ \overrightarrow{0\ \ 0} \end{matrix} \right\rangle,$$

$$L_{1110} = \left\langle \begin{matrix} \overleftarrow{1\ \ 1} \\ 1\ \ \tilde{0} \end{matrix} \right\rangle,\ L_{1011} = \left\langle \begin{matrix} 1\ \ \tilde{0} \\ \overleftarrow{1\ \ 1} \end{matrix} \right\rangle,\ L_{0001} = \left\langle \begin{matrix} \overrightarrow{0\ \ 0} \\ 0\ \ \tilde{1} \end{matrix} \right\rangle,\ L_{0100} = \left\langle \begin{matrix} 0\ \ \tilde{1} \\ \overrightarrow{0\ \ 0} \end{matrix} \right\rangle,$$

$$R_{1101} = \left\langle \begin{matrix} \overleftarrow{1\ \ 1} \\ \tilde{0}\ \ 1 \end{matrix} \right\rangle,\ R_{0111} = \left\langle \begin{matrix} \tilde{0}\ \ 1 \\ \overleftarrow{1\ \ 1} \end{matrix} \right\rangle,\ R_{0010} = \left\langle \begin{matrix} \overrightarrow{0\ \ 0} \\ \tilde{1}\ \ 0 \end{matrix} \right\rangle,\ R_{1000} = \left\langle \begin{matrix} \tilde{1}\ \ 0 \\ \overrightarrow{0\ \ 0} \end{matrix} \right\rangle.$$

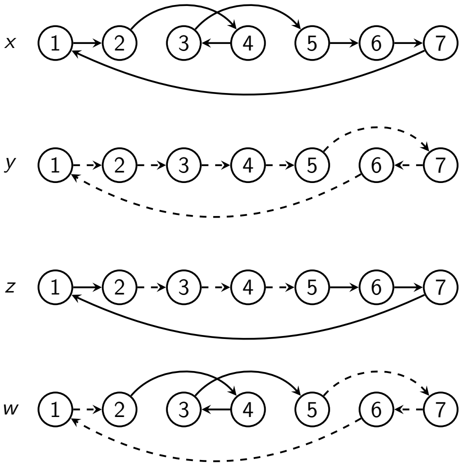
### Adjacency relation criterion

Vertices $x^v$ and $y^v$ of the polytope PSB($n$) (or PYR($n$)) are not adjacent if and only if the encodings of the tours $x$ and $y$ have a left block of the form $U, L$, have a right block of the form $U, R$, and some additional conditions are satisfied.

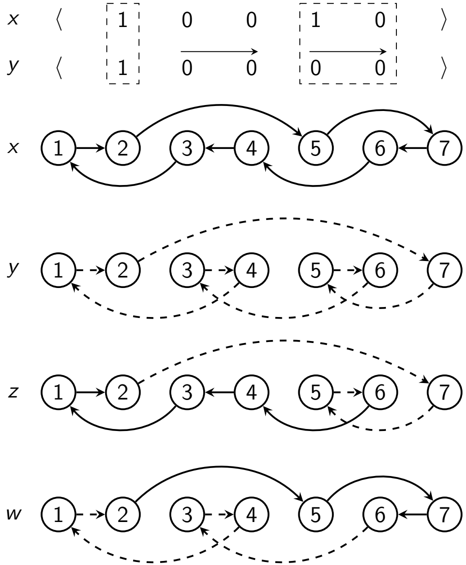
# Some examples (I)

$$x \quad \langle \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \rangle$$

$$y \quad \langle \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad \rangle$$



# Some examples (II)



**Comment**: vertex adjacency in 1-skeleton of the polytope PSB($n$) (or PYR($n$)) can be verified in linear time $O(n)$.

```
LBlock ← TRUE                                              ▷ Consider the city 1 as a left block
RBlock, zDiffx, zDiffy ← FALSE
for i ← 2 to n − 1 do
    if z (x) is different from y then                      ▷ z visits i by the edges of x
        zDiffy ← TRUE
    end if
    if LBlock = FALSE then                                 ▷ Left part
        if we found U or L block then
            LBlock ← TRUE
        end if
    end if
    if LBlock = TRUE and RBlock = FALSE then               ▷ Central part
        if z (y) is different from x in the central part then
            zDiffx ← TRUE                                  ▷ z visits i by the edges of y
        end if
        if zDiffx = TRUE and we found U or R block then
            RBlock ← TRUE                                  ▷ Go to the right part
        end if
        if the condition of the central part is violated then
            LBlock, zDiffx ← FALSE                         ▷ Go to the left part
        end if
    end if
end for
if LBlock, zDiffx, zDiffy = TRUE then                      ▷ Consider the city n as a right block
    return sufficient condition of nonadjacency is satisfied
else
    return sufficient condition of nonadjacency is not satisfied
end if
```

# 2. Graph diameter

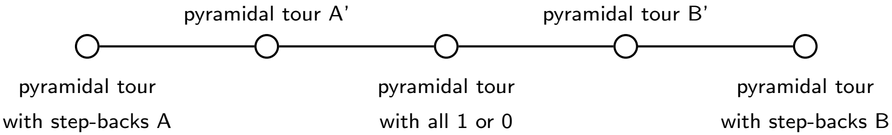
**Theorem (Bondarenko, A.N., 2018)**

The diameter of PYR($n$) 1-skeleton equals 2 for all $n \geq 6$.

**Theorem**

The diameter of the 1-skeleton of the polytope PSB($n$) is bounded above by 4.

**Idea of proof**



# 3. Clique number

### Theorem

The clique numbers of 1-dkeletons for both polytopes PYR($n$) and PSB($n$) are quadratic in the parameter $n$:

$$\omega(\mathrm{PYR}(n)) = \omega(\mathrm{PSB}(n)) = \Theta\left(n^2\right).$$

### Example of the lower bound

$$\begin{array}{cccc}
\langle 0\ 0\ 0 \mid 0\ 0\ 0 \rangle & \langle 0\ 0\ 0 \mid 0\ 0\ 1 \rangle & \langle 0\ 0\ 0 \mid 0\ 1\ 1 \rangle & \langle 0\ 0\ 0 \mid 1\ 1\ 1 \rangle \\
\langle 1\ 0\ 0 \mid 0\ 0\ 0 \rangle & \langle 1\ 0\ 0 \mid 0\ 0\ 1 \rangle & \langle 1\ 0\ 0 \mid 0\ 1\ 1 \rangle & \langle 1\ 0\ 0 \mid 1\ 1\ 1 \rangle \\
\langle 1\ 1\ 0 \mid 0\ 0\ 0 \rangle & \langle 1\ 1\ 0 \mid 0\ 0\ 1 \rangle & \langle 1\ 1\ 0 \mid 0\ 1\ 1 \rangle & \langle 1\ 1\ 0 \mid 1\ 1\ 1 \rangle \\
\langle 1\ 1\ 1 \mid 0\ 0\ 0 \rangle & \langle 1\ 1\ 1 \mid 0\ 0\ 1 \rangle & \langle 1\ 1\ 1 \mid 0\ 1\ 1 \rangle & \langle 1\ 1\ 1 \mid 1\ 1\ 1 \rangle
\end{array}$$

# Conclusion

| | Hamiltonian cycles | Pyramidal tours | Pyramidal tours with step-backs |
|---|---|---|---|
| Complexity of TSP problem | NP-hard | $O(n^2)$ | $O(n^2)$ |
| Vertex adjacency in 1-skeleton | NP-complete | $O(n)$ | $O(n)$ |
| Diameter of 1-skeleton | 2 for ATSP($n$)<br>$\leq 4$ for TSP($n$) | 2 | $\leq 4$ |
| Clique number of 1-skeleton | $\Omega\left(2^{(\sqrt{n}-9)/2}\right)$ | $\Theta\left(n^2\right)$ | $\Theta\left(n^2\right)$ |

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| Clique number of 1-skeleton | $\Omega\left(2^{\left(\sqrt{n}-9\right)/2}\right)$ | $\Theta\left(n^2\right)$ | $\Theta\left(n^2\right)$ |

**Comment:** complexity of verifying vertex adjacency in 1-skeleton here coincides with the complexity of finding a Hamiltonian decomposition of 4-regular union multigraph $x \cup y$.

Thank you for your attention!