Description of contacts in fluid-beam systems

8ECM - Portorož - MS36

Mathematical analysis : the interaction of fluids/viscoelastic materials and solids

Matthieu Hillairet
(Université de Montpellier & IUF)

Joint works with :

- J.-J. Casanova (Univ. Paris-Dauphine)
- C. Grandmont (INRIA)

Fluid/structure problem

System

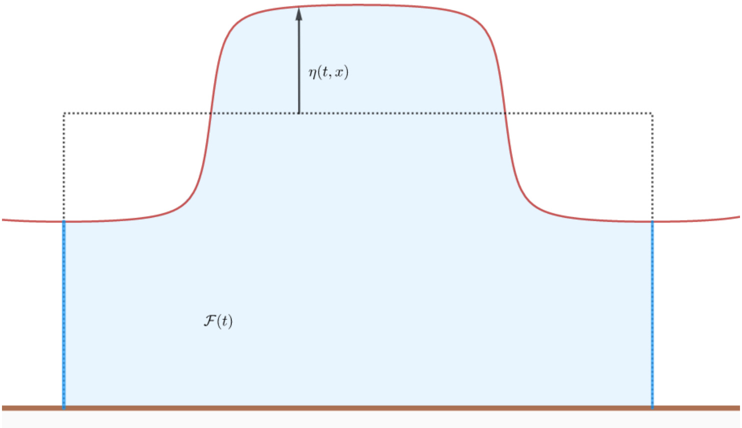

FIGURE – Film of fluid delimited by an elastic membrane

Fluid/structure problem
Model

- Fluid dynamics :

$$\text{(NS)} \qquad \left.\begin{aligned} &\rho_f\left(\partial_t u_f + u_f\cdot\nabla u_f\right) = \mu\Delta u_f - \nabla p_f \\ &\operatorname{div} u_f = 0 \end{aligned}\right\} \quad \text{on } \mathcal{F}(t)$$

- Structure dynamics : $\eta = h - 1$

$$\rho_s\partial_{tt}\eta - \beta\partial_{xx}\eta + \alpha\partial_{xxxx}\eta + \gamma\partial_{txx}\eta - \delta\partial_{ttxx}\eta = \phi \qquad \text{on } (0, L)$$

- Interface conditions :

$$\text{(Inter)} \qquad \left.\begin{aligned} &u_f(x,\eta(x,t),t) = \partial_t\eta(x,t)e_2 \\ &\phi = \sqrt{1+|\partial_x\eta|^2}\,[(2\mu D(u_f) - p_f\mathbb{I}_d)n]_{y=\eta(t,x)}\cdot e_2 \end{aligned}\right\} \quad \text{on } (0, L)$$

- Boundary conditions :

(BC) L-periodicity in x $\qquad u_f(x,0,t) = 0$ on $(0, L)$

First remarks

- Dissipative system :

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\mathcal{E}_f + \mathcal{E}_s\right] + \mathcal{D} = 0.$$

with

$$\mathcal{E}_f = \frac{1}{2}\int_{\mathcal{F}(t)} \rho_f |u|^2$$

$$\mathcal{E}_s = \frac{1}{2}\int_0^L \rho_s |\partial_t \eta|^2 + \alpha |\partial_{xx}\eta|^2 + \beta |\partial_x \eta|^2 + \delta |\partial_{tx}\eta|^2$$

$$\mathcal{D} = 2\mu \int_{\mathcal{F}(t)} |D(u)|^2 + \gamma \int_0^L |\partial_{tx}\eta|^2$$

- Invariant :

$$\int_0^L \partial_t \eta(x,t)\mathrm{d}x = 0$$

Cauchy theory before contact

2D/1D setting

- Weak solutions before collapse

  C. Grandmont '08

- Strong solutions (local-in-time) :

  H. Beirao da Veiga '04, J.-J. Casanova '17, J. Lequeurre '11,'13, C. Grandmont, J. Lequeurre & M.H. '19.

Related results

A. Chambolle et al *'05,*
C.H. Arthur Cheng, D. Coutand & S. Shkoller '07'08
B. Muha & S. Canić '13
D. Lengeler & M. Růžička '14
B. Muha & S. Schwarzacher '19, M. Badra & T. Takahashi '19

Analysis of fluid+solid problems

Cauchy theory (before contact)

T. Takahashi '03 M.D. Gunzburger, H.-C. Lee & G. A. Seregin '00,
B. Desjardins & M. Esteban 99','00.

Contact description : (non exhaustive review)

Influence of solid roughness on contact occurence :
V. Starovoitov '03, D. Gérard-Varet & M.H. '11, S. Filippas & A. Tersenov '21

Global existence of weak solutions regardless contacts (2D) :
K.H. Hoffman & V. Starovoitov '03,
J. A. San Martin, V. Starovoitov & M. Tucsnak '02

Global existence of weak solutions regardless contacts (3D) :
E. Feireisl '03

Main results

Global-in-time existence of strong solutions (dissipative beam)
[*C. Grandmont, M.H. '16*]

For arbitrary sufficiently smooth and compatible initial data $(\eta^0, \dot{\eta}^0, u^0)$ with no collapse, the maximal strong solution is global. In particular, no collapse occurs in finite time.

Global-in-time existence of weak solutions (non dissipative beam)
[*J.-J. Casanova, C. Grandmont, M.H. '21*]

For arbitrary finite-energy compatible initial data $(\eta^0, \dot{\eta}^0, u^0)$ with no collapse, there exist a global finite-energy weak solution wether collapse occurs or not.

Theorem [*C. Grandmont, M.H.* '16]

In the case $\alpha > 0, \gamma > 0$, for arbitrary compatible initial data with no collapse :

$$(\eta^0, \dot{\eta}^0, u^0) \in H^3_\sharp \times H^1_\sharp \times H^1(\mathcal{F}^0)$$

there exists a unique global-in-time (strong) solution. In particular,

$$\min_{x \in [0,L]} h(x,t) > 0 \quad \forall\, t > 0.$$

No-contact argument

- Beam/Reynolds system (on $(-\pi, \pi)$) :

$$\begin{aligned} \partial_{tt} h + \alpha \partial_{xxxx} h - \beta \partial_{xxx} h - \gamma \partial_{txx} h = q\,, \\ \mu \partial_x (h^3 \partial_x q) = \partial_t h, \end{aligned} \qquad \int_{-\pi}^{\pi} q = 0\,,$$

  with 2π-periodic b.c.

Fluid/structure interactions problem

- Energy estimate (multiply by $(\partial_t h, q)$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}(|\partial_t h|^2+\alpha|\partial_{xx}h|^2+\beta|\partial_x h|^2)\right]+\int_0^t\int_{-\pi}^{\pi}\left[\gamma|\partial_{tx}h|^2+\mu h^3|\partial_x q|^2\right]\leqslant C_0\,.$$

- Distance estimate (multiply first equation by $-\partial_{xx}h$)

$$\frac{d}{dt}\left[\int_{-\pi}^{\pi}\left(\frac{\gamma}{2}|\partial_{xx}h|^2-\partial_t h\,\partial_{xx}h\right)\right]+\int_{-\pi}^{\pi}\left(\beta|\partial_{xx}h|^2+\alpha|\partial_{xxx}h|^2-|\partial_{tx}h|^2\right)$$
$$=-\int_{-\pi}^{\pi}q\partial_{xx}h$$

- Regularity estimate (multiply first equation by $-\partial_{txx}h$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}\left(|\partial_{tx}h|^2+\alpha|\partial_{xxxx}h|^2+\beta|\partial_{xx}h|^2\right)\right]+\gamma\int_0^t\int_{-\pi}^{\pi}|\partial_{txx}h|^2\leqslant C_2\,.$$

Fluid/structure interactions problem

- Energy estimate (multiply by $(\partial_t h, q)$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}(|\partial_t h|^2+\alpha|\partial_{xx}h|^2+\beta|\partial_x h|^2)\right]+\int_0^t\int_{-\pi}^{\pi}\left[\gamma|\partial_{tx}h|^2+\mu h^3|\partial_x q|^2\right]\leqslant C_0\,.$$

- Distance estimate (multiply first equation by $-\partial_{xx}h$)

$$\frac{d}{dt}\left[\int_{-\pi}^{\pi}\left(\frac{\gamma}{2}|\partial_{xx}h|^2-\partial_t h\,\partial_{xx}h\right)\right]+\int_{-\pi}^{\pi}\left(\beta|\partial_{xx}h|^2+\alpha|\partial_{xxx}h|^2-|\partial_{tx}h|^2\right)$$
$$=-\frac{1}{2}\int_{-\pi}^{\pi}h^3\partial_x q\partial_x\left[\frac{1}{h^2}\right]$$

- Regularity estimate (multiply first equation by $-\partial_{txx}h$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}\left(|\partial_{tx}h|^2+\alpha|\partial_{xxxx}h|^2+\beta|\partial_{xx}h|^2\right)\right]+\gamma\int_0^t\int_{-\pi}^{\pi}|\partial_{txx}h|^2\leqslant C_2\,.$$

Fluid/structure interactions problem

- Energy estimate (multiply by $(\partial_t h, q)$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}(|\partial_t h|^2+\alpha|\partial_{xx}h|^2+\beta|\partial_x h|^2)\right]+\int_0^t\int_{-\pi}^{\pi}\left[\gamma|\partial_{tx}h|^2+\mu h^3|\partial_x q|^2\right]\leqslant C_0\,.$$

- Distance estimate (multiply first equation by $-\partial_{xx}h$)

$$\frac{d}{dt}\left[\int_{-\pi}^{\pi}\left(\frac{\gamma}{2}|\partial_{xx}h|^2-\partial_t h\,\partial_{xx}h\right)\right]+\int_{-\pi}^{\pi}\left(\beta|\partial_{xx}h|^2+\alpha|\partial_{xxx}h|^2-|\partial_{tx}h|^2\right)$$
$$=\frac{1}{2\mu}\int_{-\pi}^{\pi}\frac{\partial_t h}{h^2}\,.$$

- Regularity estimate (multiply first equation by $-\partial_{txx}h$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}\left(|\partial_{tx}h|^2+\alpha|\partial_{xxxx}h|^2+\beta|\partial_{xx}h|^2\right)\right]+\gamma\int_0^t\int_{-\pi}^{\pi}|\partial_{txx}h|^2\leqslant C_2\,.$$

Fluid/structure interactions problem

- Energy estimate (multiply by $(\partial_t h, q)$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}(|\partial_t h|^2+\alpha|\partial_{xx}h|^2+\beta|\partial_x h|^2)\right]+\int_0^t\int_{-\pi}^{\pi}\left[\gamma|\partial_{tx}h|^2+\mu h^3|\partial_x q|^2\right]\leqslant C_0\,.$$

- Distance estimate (multiply first equation by $-\partial_{xx}h$)

$$\int_{-\pi}^{\pi}\left(\frac{\gamma}{2}|\partial_{xx}h|^2+\frac{1}{2\mu h}-\partial_t h\,\partial_{xx}h\right)+\int_0^t\int_{-\pi}^{\pi}\left(\beta|\partial_{xx}h|^2+\alpha|\partial_{xxx}h|^2-|\partial_{tx}h|^2\right)=C_0$$

Classical Lemma.

$$\left(\|h\|_{H^2}+\|1/h\|_{L^1}\leqslant M\right)\implies\left(\|1/h\|_{L^\infty}\leqslant C(M)\right)$$

- Regularity estimate (multiply first equation by $-\partial_{txx}h$)

$$\frac{1}{2}\left[\int_{-\pi}^{\pi}\left(|\partial_{tx}h|^2+\alpha|\partial_{xxxx}h|^2+\beta|\partial_{xx}h|^2\right)\right]+\gamma\int_0^t\int_{-\pi}^{\pi}|\partial_{txx}h|^2\leqslant C_2\,.$$

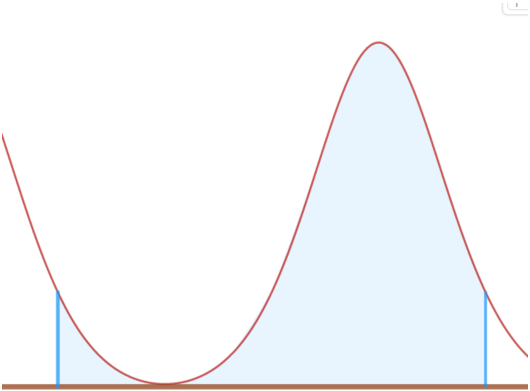

FIGURE – Contact geometry

Study of contact – non-dissipative beam

Choice of a functional framework

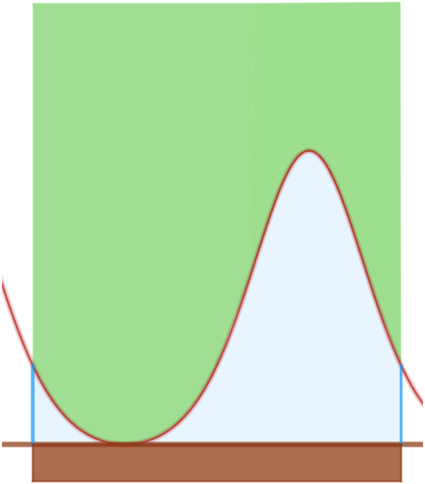

FIGURE – Extended domain

Study of contact – non-dissipative beam

Choice of a functional framework

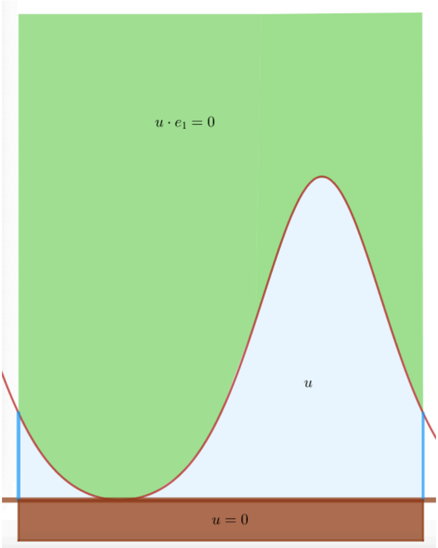

FIGURE – Extended velocity-field

Notations

Beam with no dissipation case : $\gamma = 0, \alpha > 0$.

Notations. Given a $W^{1,\infty}_\sharp$ height function s.t. $0 \leqslant h \leqslant M$:

- Unknown spaces

$$\begin{aligned} K[h] = \{ & w \in L^2_\sharp((0,L)\times(-1,M)) \text{ s.t. } \operatorname{div} w = 0 \\ & \text{and } w = 0 \text{ on } y < 0 \text{ and } w \cdot e_1 = 0 \text{ in } y > h \,\} \end{aligned}$$

$$X[h] = \{(w,d) \in K[h] \times L^2_\sharp \text{ s.t. } \int_0^L d = 0 \text{ and } w_2(\cdot, M) = d\}.$$

- Test-function spaces

$$\begin{aligned} \mathcal{K}[h] = \{ & w \in C^\infty_\sharp((0,L)\times(-1,M)) \text{ s.t. } \operatorname{div} w = 0 \\ & \text{and } w = 0 \text{ on } \mathcal{V}(y < 0) \text{ and } v \cdot e_1 = 0 \text{ in } \mathcal{V}(y > h) \,\} \end{aligned}$$

$$\mathcal{X}[h] = \{(w,d) \in \mathcal{K}[h] \times C^\infty_\sharp \text{ s.t. } \int_0^L d = 0 \text{ and } w_2(\cdot, M) = d\}.$$

Definition of weak solution

Definition Given a non-colliding initial data $(\eta^0, \dot\eta^0) \in H^2_\sharp \times L^2_\sharp$ and $u^0 \in L^2_\sharp(\mathcal{F}^0)$ we call weak solution on $(0, T)$ a pair $(\bar u, \eta)$ satisfying :

- $(\bar u, \eta) \in L^\infty(0, T; L^2_\sharp((0, L) \times (-1, M))) \times (L^\infty(0, T; H^2_\sharp) \cap W^{1,\infty}(0, T; L^2_\sharp))$ with
$$(\bar u(\cdot, t), \partial_t \eta(\cdot, t)) \in X[h(\cdot, t)] \text{ for a.e. } t \quad \nabla \bar u \in L^2(y < h).$$
- $\bar u(x, h(x, t)) = \partial_t \eta(t, x)$ for a.e. (t,x)
- for any smooth (w, d) such that $(w(\cdot, t), d(\cdot, t)) \in \mathcal{X}[h(\cdot, t)]$ for all t there holds :
$$\rho_f \left(\int_{\mathcal{F}(t)} \bar u \cdot w - \int_0^t \int_{\mathcal{F}(t)} \bar u \cdot w + \bar u \cdot \nabla w \cdot \bar u \right) + \rho_s \left(\int_0^L \partial_t \eta d - \int_0^L \partial_t \eta \partial_t d \right)$$
$$+ \mu \int_0^t \int_{\mathcal{F}(t)} \nabla \bar u : \nabla w + \alpha \int_0^t \int_0^L \partial_{xx} \eta \partial_{xx} d$$
$$= \rho_f \int_{\mathcal{F}^0} u^0 \cdot w(0) + \rho_s \int_0^L \dot\eta^0 d(0)$$

Main result

Theorem [*J.J. Casanova, C. Grandmont, M.H.* '21]
Given $T > 0$ a non-colliding initial data $(\eta^0, \dot{\eta}^0) \in H^2_\sharp \times L^2_\sharp$ and $u^0 \in L^2_\sharp(\mathcal{F}^0)$ satisfying :

$$\mathrm{div}\, u_0 = \text{ in } \mathcal{F}^0 \qquad u^0 \cdot e_2 = 0 \text{ on } y = 0$$
$$u^0 \cdot n = \dot{\eta}^0 e_2 \cdot n \text{ on } y = 1 + \eta^0,$$

there exists at least one weak solution on $(0, T)$.

Remark

- To be compared with [*J. A. San Martin, V. Starovoitov, M. Tucsnak* '02]
- No uniqueness result ! !

Roadmap of the proof

- Approximate solution sequence :
  $(u_\gamma, \eta_\gamma)_{\gamma>0}$ weak solutions to the system with extra dissipation terms.
  $$\bar{u}_\gamma = \mathbf{1}_{y>1+\eta_\gamma} \partial_t \eta_\gamma e_2 + \mathbf{1}_{0<y<1+\eta} u_\gamma .$$
- Uniform estimate :
  $$\sup_{(0,T)} \left[\int_{\mathcal{F}_\gamma(t)} \rho_f |u_\gamma|^2 + \int_0^L \rho_s |\partial_t \eta_\gamma|^2 + \alpha |\partial_{xx} \eta_\gamma|^2 \right] + \int_0^T \int_{\mathcal{F}_\gamma(t)} |\nabla u_\gamma|^2 + \gamma \int_0^L |\partial_{tx} \eta_\gamma|^2 \leqslant C_0$$
- Key issue : obtain (strong) L^2-compactness to pass to the limit in
  $$\int_0^t \int_{\mathcal{F}_{\eta_\gamma}} \bar{u}_\gamma \cdot \nabla w \cdot \bar{u}_\gamma$$

Approximation/Projection method

[J.A. San Martin, V. Starovoitov, M. Tucsnak '02]

- To prove strong convergence of $\eta_\gamma \to \eta$
- To construct positive approximations from below $(\underline{h}_\delta)_{\delta>0}$ of $1+\eta$
- To prove L^2-compactness of

$$(\mathbb{P}[\underline{h}_\delta](\bar{u}_\gamma, \partial_t \eta_\gamma))_{\gamma>0}$$

with $\mathbb{P}[\underline{h}_\delta] : L^2_\sharp((0,L)\times(0,1)) \times L^2_\sharp \to X[\underline{h}_\delta]$

- To prove uniform smallness of

$$\|\mathbb{P}[\underline{h}_\delta]](\bar{u}_\gamma, \partial_t\eta_\gamma) - (\bar{u}_\gamma, \partial_t\eta_\gamma)\|_{L^2\times L^2_\sharp}.$$

Is it worth it ?

Analysis of asymptotic problems

Example : ($\sharp =$ periodic boundary conditions)

$$\begin{aligned} \partial_t h - \partial_x(h^3 \partial_x q) &= \partial_x(h^3 \partial_x f)\,, \\ q &= (-\partial_{xx})^{1+\theta} h, \end{aligned}$$

Theorem [*J-J Casanova & M.H.* '21]

Assume $h^0 \in H_\sharp^{1+\theta}$ and

$$f \in L^2(0, T; H_\sharp^{-\theta}) \quad \partial_t f \in L^2(0, T; H_\sharp^{-(1+\theta)})$$

Then

- if $\theta > 1/2$ the solution exists on $(0, T)$ and in particular it does not vanish on $(0, T)$,
- if $\theta < 1/2$ we can construct f such that the solution exists on $(0, T)$ and contact holds in T.