

Circularly compatible ones, D-circularity, and proper circular-arc bigraphs

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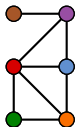
Proper interval graphs

The class of proper interval graphs is possibly the most studied subclass of interval graphs.

Proper interval graphs

A graph is a **proper interval graph** if it is possible to assign an interval to each vertex so that:

- ▶ two vertices are adjacent if and only if the corresponding intervals intersect and
- ▶ no two of these intervals are one properly contained in the other.

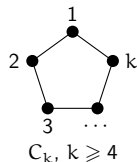
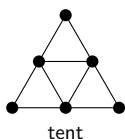
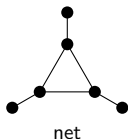
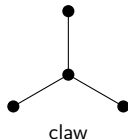


Such a set of intervals is called a **proper interval model** of the graph.

Proper interval graphs

Theorem (Wegner, 1967; Roberts, 1969)

A graph is a proper interval graph if and only if it does not contain any of the following graphs as induced subgraphs



Theorem (Looges and Olariu, 1993; Corneil et al., 1995; Hell and Huang, 2004)

There is a linear-time algorithm that, given any graph, decides whether or not it is a proper interval graph. Moreover, in the affirmative case, it returns a **proper interval model** and, in the negative case, a **minimal forbidden induced subgraph**.

Among other characterizations of proper interval graphs, there is a matrix characterization due to Roberts.

Linearly compatible ones property

In connection with such matrix characterization of proper interval graphs, Tucker (1969) introduced the linearly compatible ones property.

We say a row of a matrix is **trivial** if it has only 0's or only 1's.

Linearly compatible ones property

A matrix has the **linearly compatible ones property** if there is a permutation of its rows and columns such that:

- ▶ the 1's in each row form an interval in such a way that, excluding the trivial rows, the sequences of left and of right endpoints of these intervals are monotone non-decreasing and
- ▶ the 1's in each column also form an interval.

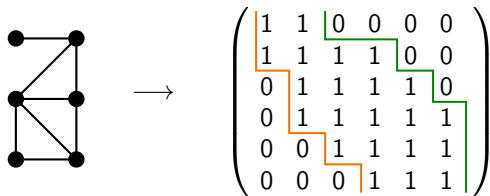
$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Linearly compatible ones and proper interval graphs

An **augmented adjacency matrix** is obtained from an adjacency matrix by adding 1's all along the main diagonal.

Theorem (Roberts, 1971)

A graph is a proper interval graph if and only if its augmented adjacency matrix has the linearly compatible ones property.



Linearly compatible ones and D-interval matrices

Interestingly, the linearly compatible ones property turns out to be equivalent to several notions that were defined in subsequent years:

- ▶ D-interval hypergraphs (Moore, 1977),
- ▶ adjacency and enclosure property (Spinrad, Brandstädt, and Stewart, 1987),
- ▶ monotone consecutive arrangements (Sen and Sanyal, 1994),
- ▶ forward convex labelings (Lai and Wei, 1997).

Linearly compatible ones and D-interval matrices

D-interval property

A matrix has the **D-interval property** if it is possible to permute its columns so that:

- ▶ the 1's in each row form an interval and
- ▶ if two such intervals are one contained in the other, then they share an endpoint.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \underline{1} & \underline{1} & 0 & 0 \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ \underline{1} & \underline{1} & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & \underline{1} & 0 \end{pmatrix}$$

Theorem (Moore, 1977; Lai and Wei, 1997; Sen and Sanyal, 1994; S., 2021)

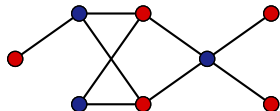
A matrix has the linearly compatible ones property if and only if it has the D-interval property.

Proper interval bigraphs

Proper interval bigraphs

A **proper interval bigraph** is a bipartite graph such that it is possible to assign an interval to each vertex so that:

- ▶ vertices on different sides of the bipartition are adjacent if and only if the corresponding intervals intersect, and
- ▶ there are no two vertices on the same side of the bipartition whose intervals are one properly contained in the other.

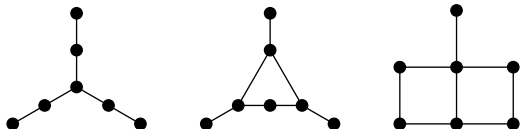


Such a set of intervals is called a **proper interval bimodel** of the graph.

Proper interval bigraphs

Theorem (Steiner, 1996)

A bipartite graph is a proper interval bigraph if and only if it contains as induced subgraphs neither C_{2k} for any $k \geq 3$ nor any of the following graphs:



Thus, proper interval bigraphs coincide with bipartite AT-free, bipartite permutation, bipartite co-comparability, bipartite tolerance, etc.

Theorem (Spinrad, Brandstädt, and Stewart, 1987; Hell and Huang, 2004)

There is a linear-time algorithm that, given any bipartite graph, decides whether or not it is a proper interval bigraph. Moreover, in the affirmative case it returns a **proper interval bimodel** and, in the negative case, it returns a **minimal forbidden induced subgraph**.

Graph classes with the linearly compatible ones property

...has the linearly
compatible ones
property

the augmented
adjacency matrix...

**proper interval
graphs¹**

the biadjacency matrix...

**proper interval
bigraphs²**

¹Roberts (1971)

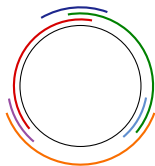
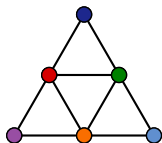
²Sen and Sanyal (1994) and S. (2021)

Proper circular-arc graphs

Proper circular-arc graphs

A graph is a **proper circular-arc graph** if it is possible to assign an arc of a circle to each vertex so that

- ▶ two vertices are adjacent if and only if the corresponding arcs intersect and
- ▶ no two such arcs are one properly contained in the other.

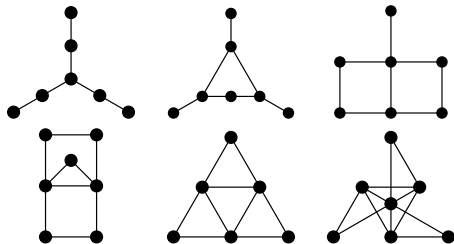


Such a set of arcs is called a **proper circular-arc model** of the graph.

Proper circular-arc graphs

Theorem (Tucker, 1974)

A graph is a proper circular-arc graph if and only if does not contain as an induced subgraph $C_k \cup K_1$ ($k \geq 4$), $\overline{C_{2k}}$ ($k \geq 3$), $\overline{C_{2k+1} \cup K_1}$ ($k \geq 1$) nor the complement of any of the following graphs:



Theorem (Deng, Hell, and Huang, 1996; Nussbaum, 2008)

There is a linear-time algorithm that, given any graph, decides whether or not it is a proper circular-arc graph. Moreover, in the affirmative case it returns a **proper circular-arc model** and, in the negative case, it returns a **minimal forbidden induced subgraph**.

Circularly compatible ones property

Tucker (1969) introduced the **circularly compatible ones property** to achieve for proper circular-arc graphs a characterization analogous to that of Roberts for proper interval graphs.

Circularly compatible ones property

A matrix has the **circularly compatible ones property** if there is a permutation of its rows and columns such that:

- ▶ the 1's in each row form a circular interval in such a way that, excluding the trivial rows, the sequences of left and of right endpoints of these circular intervals are circularly monotone, and
- ▶ the 1's in each column form a circular interval.

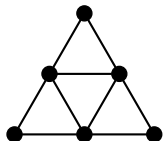
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Such a permutation of rows and columns is called a **circularly compatible ones border** of the matrix.

Proper circular-arc graphs and circularly compatible ones

Theorem (Tucker, 1969)

A graph is a proper circular-arc graph if and only if its augmented adjacency matrix has the circularly compatible ones property.



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Two problems by Tucker

Problems (Tucker, 1969)

For the circularly compatible ones property in arbitrary matrices (not restricted to augmented adjacency matrices):

- 1 find a forbidden submatrix characterization and
- 2 find an efficient recognition algorithm.

We solved both problems.

Circularly compatible ones property

Theorem (S., 2021)

A matrix has the circularly compatible ones property if and only if it does not contain as a submatrix any of the following matrices or their transposes, or permutations of rows and columns of them:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & & & 0 \\ 1 & & 1 & & 0 \\ & & & \ddots & \vdots \\ & & & & 1 & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & & & 1 \\ 0 & 0 & 0 & & \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 0 & 1 \\ 0 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}$$

Theorem (S., 2021)

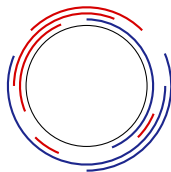
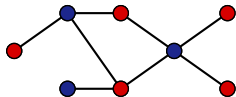
There exists a linear-time algorithm that, given any matrix, decides whether or not it has the circularly compatible ones property. Moreover, in the affirmative case it returns a **circularly compatible ones biorder** and, in the negative case, a **minimal forbidden submatrix**.

Proper circular-arc bigraphs

Proper circular-arc bigraphs

A **proper circular-arc bigraph** is a bipartite graph such that it is possible to assign an arc of a circle to each vertex so that

- ▶ vertices on different sides of the bipartition are adjacent if and only if the corresponding arcs intersect and
- ▶ there are no two vertices on the same side of the bipartition whose arcs are one properly contained in the other.



If so, the set of arcs is called a **proper circular-arc bimodel** of the graph.

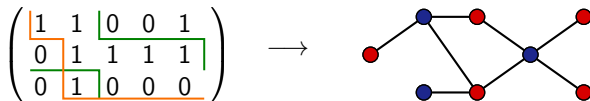
Proper circular-arc bigraphs: characterization

Basu, Das, Ghosh, and Sen (2013) characterized the subclass of proper circular-arc bigraphs having biadjacency matrices without trivial rows.

We characterized the whole class of the proper circular-arc bigraphs.

Theorem (S., 2021)

The proper circular-arc bigraphs are precisely the bipartite graphs associated with the matrices with the circularly compatible ones property.



Graph classes with the compatible ones properties

	...has the linearly compatible ones property	...has the circularly compatible ones property
the augmented adjacency matrix...	proper interval graphs¹	proper circular-arc graphs²
the biadjacency matrix...	proper interval bigraphs³	proper circular-arc bigraphs⁴

¹Roberts (1971)

²Tucker (1969)

³Sen and Sanyal (1994) and S. (2021)

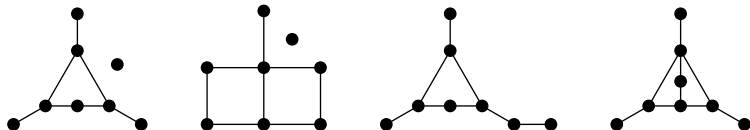
⁴S. (2021)

Proper circular-arc bigraphs: structure

By virtue of the equivalence between proper circular-arc bigraphs and the circularly compatible ones property, our characterization of the latter in terms of forbidden submatrices implies the following.

Theorem (S., 2021)

A bipartite graph is a proper circular-arc bigraph if and only if it contains as an induced subgraph neither $C_{2k} \cup K_1$ nor its bipartite complement for any $k \geq 3$, nor any of the following graphs:



Proper circular-arc bigraphs: recognition

Basu, Das, Ghosh, and Sen (2013) posed the problem of finding an efficient algorithm for recognizing proper circular-arc bigraphs. The same problem was reposed by Das and Chakraborty (2017).

Thanks to the equivalence between proper circular-arc bigraphs and the circularly compatible ones property, together with our recognition algorithm for the latter, we also obtain a solution to this problem.

Theorem (S., 2021)

There exists a linear-time algorithm that, given any bipartite graph, decides whether or not it is a proper circular-arc bigraph. Moreover, in the affirmative case it returns a **proper circular-arc bimodel** and, in the negative case, a **minimal forbidden induced subgraph**.

Interestingly, from these results for the proper circular-arc bigraphs it is possible to derive the known analogous results for the proper interval bigraphs (Steiner, 1996; Spinrad, Brandstädt, and Stewart, 1987; Hell and Huang, 2004).

D-circular property

We obtain our results for the circularly compatible property by proving analogous but more general results for the following variant of the D-interval property.

D-circular property

A matrix has the **D-circular property** if there exists some column permutation such that:

- 1 the 1's in each row form a circular interval, and
- 2 if two such circular intervals are one contained in the other, then they share an endpoint.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} & \underline{1} \end{pmatrix}$$

If so, the column permutation is called a **D-circular order** of the matrix.

D-circular property and circular-ones property

Remark

A matrix has the D-circular property if and only if the matrix $D(M)$ that arises by adding as new rows all the differences $r - s$ for each row r that dominates a row s has the **circular-ones property** (i.e., admits a column permutation such that in each row the 1's form a circular interval).

$$M \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

↓

$$D(M) \quad \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_1 - r_4 \\ r_2 - r_3 \\ r_2 - r_4 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ \underline{1} & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 \\ 0 & \underline{1} & 0 & 0 & \underline{1} \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} & \underline{1} \\ \underline{1} & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & \underline{1} & \underline{1} & 0 & 0 \end{pmatrix}$$

D-circular property: structure

Theorem (S., 2021)

A matrix has the D-circular property if and only if it does not have as a submatrix any of the following matrices up to permutations of rows and columns:

$$\begin{array}{ccccc} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 1 & & & 0 \\ 1 & & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & & & 1 \\ 0 & 0 & & & 1 \\ & & \ddots & & \vdots \\ & & & 0 & 0 & 1 \\ 0 & 1 & \dots & 1 & 0 & 1 \end{pmatrix} \end{array}$$

The proof relies on the preceding remark and on a characterization by forbidden submatrices for the circular-ones property from an earlier work (Safe, 2020).

D-circular property: recognition

Theorem (S., 2021)

There exists a linear-time algorithm that, given any matrix, decides whether or not it has the D-circular property. Moreover, if yes, it returns a **D-circular order** and, if no, it returns a **minimal forbidden submatrix**.

- ▶ The result does not follow by applying to $D(M)$ a linear-time algorithm for recognizing the circular-ones property because the size of $D(M)$ may be quadratic in the size of M .
- ▶ We introduce a matrix $\Delta(M)$, whose size is linear in the size of M if M has the D-circular property, and prove that M has the D-circular property if and only if $\Delta(M)$ has the circular-ones property.
- ▶ Moreover, the structural result on the previous slide implies that the D-circularity of M is equivalent to M having the circular-ones property plus M not containing one of the 13 sporadic matrices.
- ▶ The algorithm follows by relying on a certifying linear-time recognition algorithm for the circular-ones property (Safe, 2020) and adapting an algorithmic technique by Lindzey and McConnell (2016) for detecting sporadic submatrices for the consecutive-ones property.

How is D-circularity related to circularly compatible ones?

Theorem (S., 2021)

For each matrix M , the following assertions are equivalent:

- 1 M has the circularly compatible ones property.
- 2 Both M and its transpose have the D-circular property.

Hence, the structural and algorithmic results for the D-circular property translate into the analogous results the circularly compatible ones property we have presented.









Open problems

- 1 Characterize the whole class of circular-arc graphs by forbidden structures that can be found in less than $O(n^3)$ time.







The $O(n^3)$ -time bound is matched by an algorithm by Francis, Hell, and Stacho (2015).

- 2 Characterize by class of interval bigraphs by forbidden structures. Partial results were obtained by Das, Das, and Sen (2016).






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Thank you very much for your attention!

