NON-LOCAL ODE IN CONFORMAL GEOMETRY

Motivation: radially symmetric solutions for $-\Delta u = u^p$

s.t. $u(r) \to \infty$ as $r \to 0$.

$p = \frac{n+2}{n-2}$

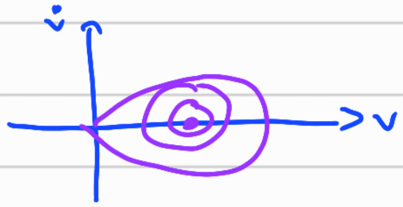
Well known: $u(r) \approx r^{-\frac{n-2}{2}}$

Change variables: $r = e^{-t}$, $u(r) = r^{-\frac{n-2}{2}} v(t)$, $t \in \mathbb{R}$.

autonomous

$$u_{rr} + \frac{n-1}{r} u_r = u^p \longleftrightarrow \boxed{-\ddot{v} + cv = cv^p} \longleftrightarrow$$

(P_1 = $-\ddot{v} + cv$)



NON-LOCAL ODE $n > 2\gamma$, $\gamma \in (0,1)$

Problem: radial solutions for

$$\boxed{(-\Delta)^\gamma u = u^p, \quad u(r) \to \infty \text{ as } r \to 0}$$

$\longrightarrow p = \frac{n+2\gamma}{n-2\gamma}$

Change variables: $r = e^{-t}$

$u(r) = r^{-\frac{n-2\gamma}{2}} v(t)$, $t \in \mathbb{R}$

We obtain:

$$\boxed{P_\gamma v = v^p}$$

"autonomous"

P_γ = conformal fractional Laplacian on the cylinder

Prop (DelaTorre - Del Pino - G. - Wei, 2017)

$\exists$ periodic solutions.

Proof: variational.

THE CONFORMAL FRACTIONAL LAPLACIAN ON THE CYLINDER.

Metric: $$\frac{|dx|^2}{r^2} = \frac{dr^2 + r^2 d\theta^2}{r^2} = \boxed{dt^2 + d\theta^2}, \quad t \in \mathbb{R}, \ \theta \in S^{n-1}$$

($r = e^{-t}$; cylindrical metric)

The operator:

$$P_\gamma v := r^{-\frac{n+2\gamma}{2}} (-\Delta)^\gamma \left(r^{\frac{n-2\gamma}{2}} v \right)$$

Characterization:

1. Via extension:

"Dirichlet-to-Neumann" operator for the Caffarelli-Silvestre extension on AdS

Spherical harmonic decomposition of $\mathbb{S}^{n-1}$:

$$-\Delta_{\mathbb{S}^{n-1}} e_m = \mu_m e_m \longrightarrow \{\mu_m\}, \{e_m\}$$

Any function on the cylinder $\mathbb{R}\times\mathbb{S}^{n-1}$ can be decomposed as $u(t,\theta) = \sum_m u_m(t)\, e_m(\theta)$.

Theorem (DelaTorre-G.)

P_γ diagonalizes $\rightsquigarrow$ $P_\gamma^{(m)}$

And if we take Fourier transform in $t \rightsquigarrow \xi$,

$$\widehat{P_\gamma^{(m)} u_m}(\xi) = \boxed{\Theta_\gamma^{(m)}(\xi)}\, \widehat{u_m}(\xi)$$

symbol

where

$$\Theta_\gamma^{(m)}(\xi) = 2^{2\gamma} \frac{\left|\Gamma\left(\frac{n}{4}+\frac{\gamma}{2}+\frac{m}{2}+\frac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(\frac{n}{4}-\frac{\gamma}{2}+\frac{m}{2}+\frac{\xi}{2}i\right)\right|^2}.$$

(Look at $m=0$)
↓
Drop superindex

Properties:

- Meromorphic in $z \in \mathbb{C}$.
- $\Theta_\gamma^{(0)}(\xi) \approx |\xi|^{2\gamma}$ as $|\xi| \to \infty$.

2. Via Mellin transform (Chan-Fontelos-G.-Wei)

$$\mathcal{M} u^{i\lambda+\alpha}(\lambda) = \int_0^\infty r^{i\lambda+\alpha}\, u(r)\,dr$$

Properties:

- $\alpha = -1$: $\mathcal{M} u(\lambda) = \left(\mathcal{F} u(e^{-t})\right)(\lambda)$
- $\alpha = -1/2$: Isometry in $L^2(0,\infty)$.

To prove: $\mathcal{M}(P_\gamma v)(\lambda) = \boxed{\Theta_\gamma(\lambda)}\, \mathcal{M}v(\lambda)$.

Recall $P_\gamma v = r^{\frac{n+2\gamma}{2}} (-\Delta)^\gamma \left(r^{-\frac{n-2\gamma}{2}} v\right)$

Calculate

$$\mathcal{M}\left(r^{\frac{n+2\gamma}{2}} \mathcal{F}^{-1}\left(|k|^{2\gamma} \mathcal{F}\left(r^{-\frac{n-2\gamma}{2}} v\right)\right)\right)$$

Use:

- $\mathcal{F} u(k) = k^{\frac{2-n}{2}} \int_0^\infty J_{\frac{n-2}{2}}(kr)\, r^{\frac{n}{2}} u(r)\, dr \rightarrow$ Hankel
- $\int_0^\infty r^\mu J_\nu(ar)\, dr = 2^\mu a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) / \Gamma\left(\frac{1-\mu+\nu}{2}\right).$

3. VIA INTEGRAL KERNEL:

$$P_\gamma v(t) = \int_{\mathbb{R}} K(t-\tilde{t})\left(v(t)-v(\tilde{t})\right) d\tilde{t} + c v(t)$$

where

$$K(t) \approx \begin{cases} |t|^{-1-2\gamma} & \text{si } |t| \to 0 \\ e^{-c|t|} & \text{si } |t| \to \infty. \end{cases}$$

ODE TYPE ARGUMENTS FOR P_γ:

(Ao - Chan - DelaTorre - Fontelos - G. - Wei, Duke 2019).

Solve $P_\gamma \varphi - \kappa \varphi = h$, $\varphi = \varphi(t)$

Take Fourier transform in t: $[\Theta_\gamma(\xi) - \kappa]\hat{\varphi} = \hat{h}$

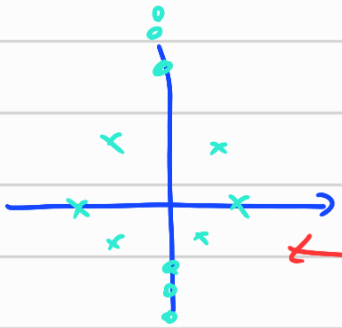
Invert: $\hat{\varphi} = \frac{1}{\Theta_\gamma(\xi) - \kappa} \hat{h}$

Fourier back:

$$\varphi(t) = \int_{\mathbb{R}} e^{-it\xi} \frac{1}{\Theta_\gamma(\xi)-\kappa} \hat{h}(\xi)\, d\xi = \int_{\mathbb{R}} e^{-it\xi} \frac{1}{\Theta_\gamma(\xi)-\kappa} \int_{\mathbb{R}} e^{i\xi\tilde{t}} h(\tilde{t})\, d\tilde{t}\, d\xi$$

$$= \int_{\mathbb{R}} G(t-\tilde{t})\, h(\tilde{t})\, d\tilde{t} \longrightarrow \text{Green's formula}$$

Problems if $\Theta_\gamma(z) - \kappa = 0$, $z \in \mathbb{C}$!



Inditial roots

$$G(t) = \sum_j c_j e^{-\sigma_j t} \quad (\sin/\cos)$$

Interesting consequence:

$$\boxed{\varphi(t) = \int_{\mathbb{R}} G(t-\tilde{t})\, h(\tilde{t})\, d\tilde{t}} = \sum_j c_j \int e^{-\sigma_j |t-\tilde{t}|} h(\tilde{t})\, d\tilde{t} =: \sum c_j \varphi_j$$

Observe that φ_j is the solution of the 2nd order ODE

$$\varphi_j'' - \sigma_j^2 \varphi_j = -2\sigma_j h$$

In summary:

non-local ODE $\longleftrightarrow$ infinite system of coupled 2nd order ODE's.

- Wronskian
- Hamiltonian ...

Applications:

1) Symmetry / symmetry breaking / non-degeneracy / uniqueness for the fractional Caffarelli-Kohn-Nirenberg inequality (Ao - DelaTorre - G.)

local $\longrightarrow$ $c \int \frac{|u|^p}{|x|^{\beta p}} \leq \int \frac{|\nabla u|^2}{|x|^{2\alpha}}$, $p = \frac{2n}{n-2+2(\beta-\alpha)}$

non-local $\longrightarrow$ $c \int \frac{|u|^p}{|x|^{\beta p}} \leq \iint \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma} |x|^{\alpha} |y|^{\alpha}} dx dy$, $p = \frac{2n}{n-2\gamma+2(\beta-\alpha)}$

2) NON-LOCAL EQUATIONS WITH DRIFT.

Regularity theory for $(-\Delta)^\gamma u + x \cdot \nabla u - \frac{\kappa}{|x|^{2\gamma}} u = f$ in $\mathbb{R}^n$

(radial solutions)

Define $e_\gamma(x) = \int_{\mathbb{R}^n} e^{-ix\xi} e^{-|\xi|^{2\gamma}/2\gamma} dx$

Theorem: Under the Fredholm condition $\int_{\mathbb{R}^n} f(x)\, e_\gamma(x)\, dx = 0$, we have

$$\|u\|_{H^{\alpha+2r}} \leq c\, \|f\|_{H^\alpha}. \qquad s \leq 1/2$$

Proof:

k=0 Take Fourier transform: $\rho = |\xi|$, $v(\rho) = \hat{u}(\xi)$.

$$(\rho^{2r} - n)v - \rho v' = \hat{f}$$

Integrate the ODE:

$$v(\rho) = \rho^{-n} e^{-\rho^{2r}/2r} \int_\rho^\infty \hat{f}(s)\, s^{n-1} e^{-s^{2r}/2r}\, ds$$

Estimate ∎

k < C_H Use cylindrical coordinates, take Mellin transform:

$$[\Theta_\gamma(\lambda) - k]\, v(\lambda) - \frac{n+2r+2i\lambda}{4s}\, v(\lambda - 2si) = \tilde{f}(\lambda)$$

Change (k=0)

$$w(\lambda i - 1) = \frac{2^{\lambda i}}{\Gamma\left(\frac{n+2r+2i\lambda}{4\gamma}\right)} \frac{\Gamma\left(\frac{n}{4} + \frac{\gamma}{2} + \frac{\lambda}{2} i\right)}{\Gamma\left(\frac{n}{4} - \frac{\gamma}{2} + \frac{\lambda}{2} i\right)}\, v(\lambda i - 1)$$

To arrive:

$$w(\lambda) - w(\lambda - 2si) =: H(\lambda)$$

Riemann-Hilbert

Take log:

Cauchy integral: $w(\lambda) = \ldots$

Modifications for $k \neq 0$: Introduce a factor $\frac{\Theta}{\Theta - k}$