# Approximate Calculation of Triple Integrals of Rapidly Oscillating Functions using Different Types of Information about Functions

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## INTRODUCTION

- The integrals of highly oscillating functions of many variables are one of the central concepts of digital signal and image processing.
- Nowadays, **methods for digital signal and image processing** are **widely used** in scientific and technical areas.
- **Current stage** of research in

  astronomy,

  radiology,

  computed tomography,

  holography and radar

  **is characterized by broad use of digital technologies**, **algorithms and methods**.
- Correspondingly, an **issue of development new** or improvement of known **mathematical models arose**, especially for new types of input information.

# INTRODUCTION

- There are the cases when **input information** about function is given:

  on the **set of traces of the function on planes**;

  **set of traces of the function on lines**;

  and **set of values of the function in the points**.

- The **report is dedicated** to the improvement of mathematical models of digital signal processing and imaging by the example of **constructing formulas of approximate calculation of integrals of highly oscillating functions of two and three variables.**

- **Oscillatory integrals** are **present in various applications**, but **it is difficult to compute them** by standard methods.
- There is a great number of articles where problems of highly-oscillating functions in regular case are discussed.
- The oldest paper has been published by L.N.G.Filon (1928);

  also it is necessary to mention Y.L.Luke (1954);

  I.Zamfirescu (1963);

  N. Bakhvalov & L. Vasileva (1968)

  D.Levin (1982), etc.

# INTRODUCTION

- The **full analysis of these methods is presented by A. Iserles** in

  *Iserles, A. On the numerical quadrature of highly–oscillating integrals I: Fourier transforms. IMA J. Numer. Anal. 24, 365–391 (2004)*

  J. *Gao and A. Iserles, Error analysis of the extended Filon-type method for highly oscillatory integrals*. Tech. Re-ports Numerical Analysis (NA2016/03) DAMPT: University of Cambridge, 2016.

- **by V. Milovanovic** in

  *Milovanovic G. V., Stanic M.P., Numerical Integration of Highly Oscillating Functions. Analytic Number Theory, Approximation Theory, and Special Functions, 2014, pp. 613-649.*

**Two and three dimensional methods** for computation of integrals of highly oscillating functions in regular case are **also discussed** in various papers. One dimensional methods are extended to the multidimensional integration.

# INTRODUCTION

- The **full analysis of these methods is presented by A. Iserles** in

  *Iserles, A., S. P. Norsett, From high oscillation to rapid approximation III: Multivariate expansions // Tech. Reports Numerical Analysis (NA2007/01)/ DAMPT – University of Cambridge. – 37 p.*

- by **V.K. Zadiraka** in

  *I. V. Sergienko, V.K. Zadiraka, O. M. Lytvyn, S.S. Melnikova, O.P. Nechuyviter, Optimal Algorithms of Calculation of Highly Oscillatory Integrals and their Applications. Monography, Tom 1. Algorithms, Kiev, 2011, p. 447.*

In this work the **problem of computing rapidly oscillating integrals** of differentiable functions **using various information operators** is considered.

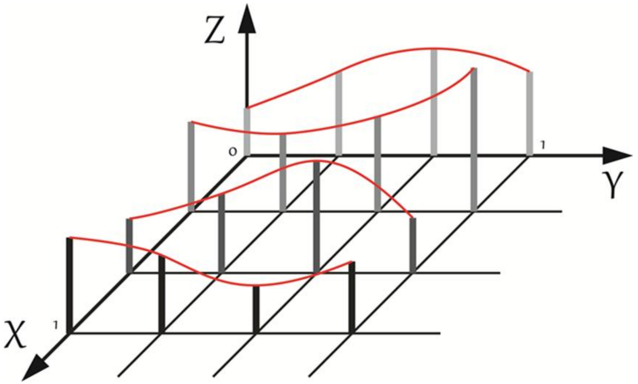
# TWO DIMENSIONAL CASE

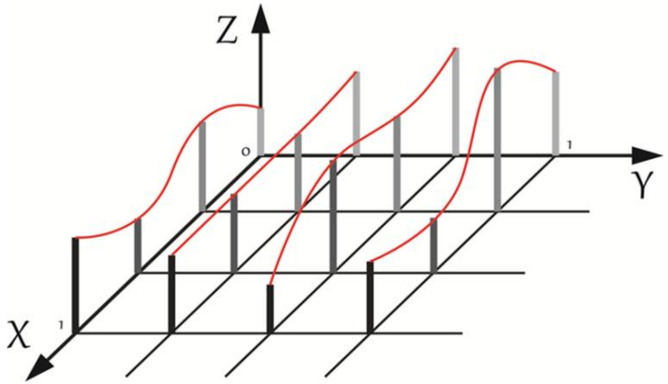
- At the beginning in this report **we present formulas** of the evaluating of **two dimensions of Fourier coefficients** with using **piece-wise splines**.

- These **formulas** are constructing in **two cases**: **input information** about function **is a set of traces of function on lines** and a **set of values of the function in the points**.

- The **main advantages** of methods are **high exactness of approximation** and **less amount of information** about function.

**Definition 1.** Under the traces of function $f(x,y)$ on the lines $x_k = k\Delta - \Delta/2$, $y_j = j\Delta - \Delta/2$, $k,j=\overline{1,\ell}$, $\Delta = 1/\ell$ we understand a function of one variable $f(x_k, y)$, $0 \le y \le 1$ or $f(x, y_j)$, $0 \le x \le 1$.



# TWO DIMENSIONAL CASE

Let

$$Jf(x,y)=\sum_{k=1}^{\ell} f(x_k,y)h_{0k}(x)+\sum_{j=1}^{\ell} f(x,y_j)H_{0j}(y)-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f(x_k,y_j)h_{0k}(x)H_{0j}(y), \qquad (1)$$

$$h_{0k}(x)=\begin{cases}1, x\in X_k,\\ 0, x\notin X_k,\end{cases} \quad k=\overline{1,\ell}, \qquad H_{0j}(y)=\begin{cases}1, y\in Y_j,\\ 0, y\notin Y_j,\end{cases} \quad j=\overline{1,\ell},$$

$$X_k=\left[x_{k-1/2},x_{k+1/2}\right],\ Y_j=\left[y_{j-1/2},y_{j+1/2}\right],$$

$$x_k=k\Delta-\Delta/2, \qquad y_j=j\Delta-\Delta/2, \qquad k,j=\overline{1,\ell},\ \Delta=1/\ell.$$

We consider $H^{2,r}(M,\widetilde{M}), r\geq 0$ – the class of functions, which are defined on the domain $G=[0,1]^2$ and

$$\left|f^{(r,0)}(x,y)\right|\leq M, \quad \left|f^{(0,r)}(x,y)\right|\leq M,\ r\neq 0, \left|f^{(r,r)}(x,y)\right|\leq \widetilde{M}, \quad r\geq 0.$$

# TWO DIMENSIONAL CASE

For numerical calculating of two-dimensional Fourier coefficients

$$I_1^2(m,n)=\int_0^1\int_0^1 f(x,y)\sin 2\pi mx\sin 2\pi ny\,dxdy \tag{2}$$

we suggest formula

$$\Phi_1^2(m,n)=\int_0^1\int_0^1 Jf(x,y)\sin 2\pi mx\sin 2\pi ny\,dxdy\,, \tag{3}$$

$$\Phi_1^2(m,n)=\sum_{k=1}^{\ell}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx\,dx\int_0^1 f(x_k,y)\sin 2\pi ny\,dy+$$

$$+\sum_{j=1}^{\ell}\int_0^1 f(x,y_j)\sin 2\pi mx\,dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny\,dy-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f(x_k,y_j)\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx\,dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny\,dy. \tag{4}$$

# TWO DIMENSIONAL CASE

**Theorem 1.** Suppose that $f(x,y) \in H^{2,1}\left(M, \widetilde{M}\right)$. Let functions $f(x,y)$ be defined by $N = 2\ell$ traces $f(x_k, y)$, $k = \overline{1,\ell}$ , $f\left(x, y_j\right)$, $j = \overline{1,\ell}$ on the system of perpendicular lines in domain $G = [0,1]^2$. It is true that

$$\rho\left(I_1^2(m,n), \Phi_1^2(m,n)\right) = \left|I_1^2(m,n) - \Phi_1^2(m,n)\right| \le \frac{\widetilde{M}}{16\ell^2} = \frac{\widetilde{M}}{4N^2}.$$

- We also can built **cubature formula** for calculating of **two-dimensional Fourier coefficients** in case when **information about** *f(x,y)* is a **set of values of the function in the points**.
- The **main advantages** of this formula are **high exactness of approximation**, the **opportunity to decrease the amount of information about function** during the calculation.

# TWO DIMENSIONAL CASE

Let
$$\tilde{J}f(x,y)=\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^2} f\left(x_k,\tilde{y}_{\tilde{j}}\right)h_{0k}(x)\tilde{H}_{0\tilde{j}}(y)+\sum_{j=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^2} f\left(\tilde{x}_{\tilde{k}},y_j\right)\tilde{h}_{0\tilde{k}}(x)H_{0j}(y)-$$
$$-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f\left(x_k,y_j\right)h_{0k}(x)H_{0j}(y), \tag{5}$$

$$X_k=\left[x_{k-1/2},x_{k+1.2}\right],\ Y_j=\left[y_{j-1/2},y_{j+1/2}\right],\quad \tilde{X}_{\tilde{k}}=\left[\tilde{x}_{\tilde{k}-1/2},\tilde{x}_{\tilde{k}+1/2}\right],\ \tilde{Y}_{\tilde{j}}=\left[\tilde{y}_{\tilde{j}-1/2},\tilde{y}_{\tilde{j}+1/2}\right],$$

$$h_{0k}(x)=\begin{cases}1, x\in X_k,\\ 0, x\notin X_k,\end{cases}\ k=\overline{1,\ell},\qquad H_{0j}(y)=\begin{cases}1, y\in Y_j,\\ 0, y\notin Y_j,\end{cases}\ j=\overline{1,\ell},$$

$$\tilde{h}_{0\tilde{k}}(x)=\begin{cases}1, x\in \tilde{X}_{\tilde{k}},\\ 0, x\notin \tilde{X}_{\tilde{k}},\end{cases}\ \tilde{k}=\overline{1,\ell^2},\qquad \tilde{H}_{0\tilde{j}}(y)=\begin{cases}1, y\in \tilde{Y}_{\tilde{j}},\\ 0, y\notin \tilde{Y}_{\tilde{j}},\end{cases}\ \tilde{j}=\overline{1,\ell^2},$$

$$x_k=k\Delta-\frac{\Delta}{2},\ y_j=j\Delta-\frac{\Delta}{2},\ k,j=\overline{1,\ell},\quad \Delta=\frac{1}{\ell},\ \tilde{x}_{\tilde{k}}=\tilde{k}\Delta_1-\frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}}=\tilde{j}\Delta_1-\frac{\Delta_1}{2},\ \tilde{k},\tilde{j}=\overline{1,\ell^2},\quad \Delta_1=\frac{1}{\ell^2}.$$

For numerical calculating of two-dimensional Fourier coefficients

$$I_1^2(m,n) = \int_0^1 \int_0^1 f(x,y) \sin 2\pi m x \sin 2\pi n y \, dx dy$$

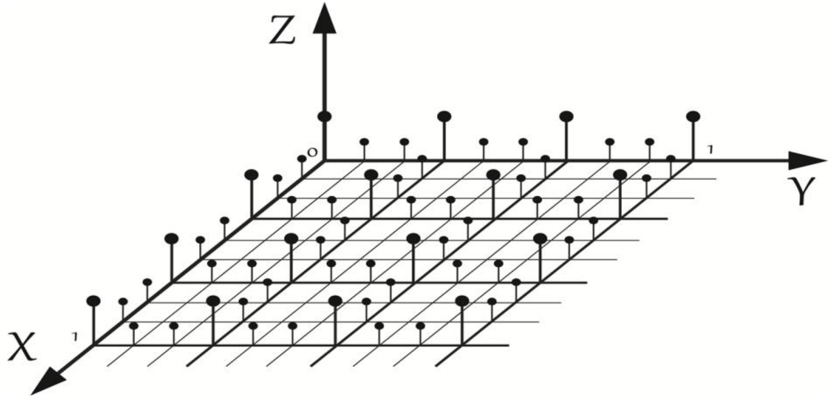
we suggest formula

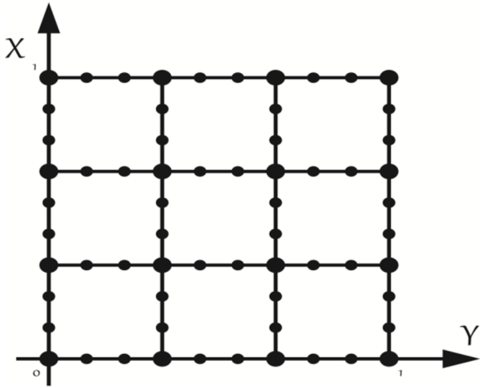
$$\widetilde{\Phi}_1^2(m,n) = \int_0^1 \int_0^1 \tilde{J} f(x,y) \sin 2\pi m x \sin 2\pi n y \, dx dy, \tag{6}$$

$$\widetilde{\Phi}_1^2(m,n) = \sum_{k=1}^{\ell} \sum_{\tilde{j}=1}^{\ell^2} f(x_k, \tilde{y}_{\tilde{j}}) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \sin 2\pi m x dx \int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}} \sin 2\pi n y dy +$$

$$+ \sum_{j=1}^{\ell} \sum_{\tilde{k}=1}^{\ell^2} f(\tilde{x}_{\tilde{k}}, y_j) \int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}} \sin 2\pi m x dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \sin 2\pi n y dy - \sum_{k=1}^{\ell} \sum_{j=1}^{\ell} f(x_k, y_j) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \sin 2\pi m x dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \sin 2\pi n y dy. \tag{7}$$

**Theorem 2.** Suppose that $f(x,y)\in H^{2,1}\left(M,\widetilde{M}\right)$. Let functions $f(x,y)$ be defined by $f(x_k,\tilde{y}_{\tilde{j}})$, $f(\tilde{x}_{\tilde{k}},y_j)$, $k,j=\overline{1,\ell}$, $\tilde{k},\tilde{j}=\overline{1,\ell^2}$ knots on domain $G=[0,1]^2$. It is true that $\rho\left(I_1^2(m,n),\widetilde{\Phi}_1^2(m,n)\right)\le\frac{M}{2\ell^2}+\frac{\widetilde{M}}{16\ell^2}=O\left(\frac{1}{\sqrt{N}}\right)$, $N=\ell^4$.



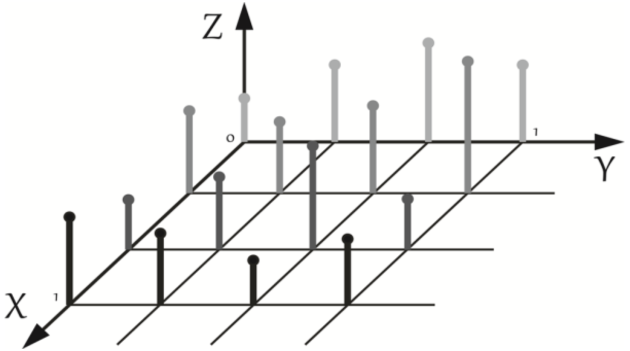



# TWO DIMENSIONAL CASE

We compare cubature formula with formula

$$\widehat{\Phi}_1^2(m,n)=\sum_{k=1}^{\ell^2}\sum_{j=1}^{\ell^2} f(x_k,y_j)\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny dy. \qquad (8)$$

It is necessary to note that $\rho\left(I_1^2(m,n),\widehat{\Phi}_1^2(m,n)\right)\le O\left(\frac{1}{\sqrt{N}}\right),\quad N=\ell^4.$



## TWO DIMENSIONAL CASE

We look for the difference between such characteristics:

- the amount of values of the function in the points $Q$ for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{Q}$ for $\widehat{\Phi}_1^2(m,n)$;
- time spent $T$ for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{T}$ for $\widehat{\Phi}_1^2(m,n)$;
- the amount of memory $P$ for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{P}$ for $\widehat{\Phi}_1^2(m,n)$ in computing.

**Example**

Calculations were done for function $f(x,y)=\sin(x+y)$ in Wolfram Mathematica 8.

# TWO DIMENSIONAL CASE

*Table 1.*

| $m$ | $n$ | $\ell$ | $\varepsilon_1$ | $Q = 2\ell^3 - \ell^2$ | $\varepsilon_2$ | $\hat{Q} = \ell^4$ |
|---|---|---|---|---|---|---|
| 4 | 4 | 10 | $1.01 \cdot 10^{-8}$ | 1900 | $1.01 \cdot 10^{-8}$ | 10000 |
| | | 25 | $2.66 \cdot 10^{-10}$ | 30625 | $2.62 \cdot 10^{-10}$ | 390625 |
| 5 | 5 | 25 | $1.69 \cdot 10^{-10}$ | 30625 | $1.67 \cdot 10^{-10}$ | 390625 |
| | | 35 | $4.43 \cdot 10^{-11}$ | 84525 | $4.36 \cdot 10^{-11}$ | 1500625 |
| 5 | 6 | 20 | $3.43 \cdot 10^{-10}$ | 15600 | $3.40 \cdot 10^{-10}$ | 160000 |
| | | 30 | $6.83 \cdot 10^{-11}$ | 53100 | $6.73 \cdot 10^{-11}$ | 810000 |
| | | 40 | $2.16 \cdot 10^{-11}$ | 126400 | $2.12 \cdot 10^{-11}$ | 2560000 |

# TWO DIMENSIONAL CASE

*Table 2.*

| $m$ | $n$ | $\ell$ | $T$, c | $\hat{T}$, c | P, б | $\hat{P}$, б |
|---|---|---|---|---|---|---|
| 4 | 4 | 10 | 0.4 | 2.0 | 4010868 | 40311244 |
| | | 25 | 8.4 | 91.1 | 39979828 | 48502900 |
| 5 | 5 | 25 | 7.1 | 84.7 | 42660708 | 42863484 |
| | | 35 | 21.6 | 337.9 | 43036868 | 43188236 |
| 5 | 6 | 20 | 3.3 | 24.2 | 44500004 | 44659660 |
| | | 30 | 8.4 | 165.0 | 44843860 | 44991564 |
| | | 40 | 13.4 | 465.5 | 45150228 | 45416260 |

**The research is dedicated to** the improvement of mathematical models of digital signal processing and imaging by the example of **constructing cubature formulas** of approximate calculation **of integrals**

$$I_1^3(m,n,p)=\int_0^1\int_0^1\int_0^1 f(x,y,z)\sin 2\pi mx \sin 2\pi ny \sin 2\pi pz\,dxdydz$$

**of highly oscillatory functions of three variables**.

The feature of the proposed cubature formulas is using the input information about function as

- a set of traces of function on planes;
- a set of traces of function on lines;
- a set of values of the function in the points.

**Definition 2.** The traces of function of three variables on the planes

$$x_k = k\Delta - \Delta/2, \quad y_j = j\Delta - \Delta/2, \quad z_s = s\Delta - \Delta/2, \quad k, j, s = \overline{1,\ell}, \quad \Delta = 1/\ell$$

are understood as function of two variables

$$f(x_k, y, z),\ 0 \le y \le 1, \quad 0 \le z \le 1, \quad f(x, y_j, z),\ 0 \le x \le 1, \quad 0 \le z \le 1, \quad f(x, y, z_s),\ 0 \le x \le 1, \quad 0 \le y \le 1.$$

**Definition 3.** The traces of function of three variables on the lines

$$\left\{(x, y, z):\ x = x_k, y = y_j, \quad x_k = k\Delta - \frac{\Delta}{2},\ y_j = j\Delta - \frac{\Delta}{2}, \Delta = \frac{1}{\ell},\ k, j = \overline{1,\ell}, \qquad 0 \le z \le 1\right\}$$

are understood as function of one variables

$$f(x_k, y_j, z), \quad 0 \le z \le 1.$$

**1 case:** the input information about function is a set of traces of function on planes

Let

$$J_1 f(x,y,z) = \sum_{k=1}^{\ell} f(x_k, y, z) h_{1k}^0(x), \quad J_2 f(x,y,z) = \sum_{j=1}^{\ell} f(x, y_j, z) h_{2j}^0(y), \tag{9}$$

$$J_3 f(x,y,z) = \sum_{s=1}^{\ell} f(x, y, z_s) h_{3s}^0(z),$$

$$X_k = [x_{k-1/2}, x_{k+1/2}], \; Y_j = [y_{j-1/2}, y_{j+1/2}], \quad Z_s = [z_{s-1/2}, z_{s+1/2}],$$

$$h_{1k}^0(x) = \begin{cases} 1, x \in X_k, \\ 0, x \notin X_k, \end{cases} \quad h_{2j}^0(y) = \begin{cases} 1, y \in Y_j, \\ 0, y \notin Y_j, \end{cases} \quad h_{3s}^0(z) = \begin{cases} 1, z \in Z_s, \\ 0, z \notin Z_s, \end{cases}$$

$$x_k = k\Delta - \frac{\Delta}{2}, \; y_j = j\Delta - \frac{\Delta}{2}, \; z_s = s\Delta - \frac{\Delta}{2}, \quad \Delta = \frac{1}{\ell}, \; k, j, s = \overline{1, \ell}.$$

**1 case:** the input information about function is a set of traces of function on planes

Operator

$$Jf(x,y,z)=J_1f(x,y,z)+J_2f(x,y,z)+J_3f(x,y,z)-$$
$$-J_1J_2f(x,y,z)-J_2J_3f(x,y,z)-J_1J_3f(x,y,z)+J_1J_2J_3f(x,y,z) \quad (10)$$

has properties:

$$Jf(x_k,y,z)=f(x_k,y,z),\quad k=\overline{1,\ell},\quad Jf(x,y_j,z)=f(x,y_j,z),\quad j=\overline{1,\ell},$$
$$Jf(x,y,z_s)=f(x,y,z_s),\quad s=\overline{1,\ell}. \quad (11)$$

**1 case:** the input information about function is a set of traces of function on planes

Let

$$H_1^{3,1}\left(M,\widetilde{M}\right)=\left\{\ f(x,y,z):\left|f^{(1,0,0)}(x,y,z)\right|\le M,\ \left|f^{(0,1,0)}(x,y,z)\right|\le M,\ \left|f^{(0,0,1)}(x,y,z)\right|\le M,\ \left|f^{(1,1,1)}(x,y,z)\right|\le \widetilde{M}\ \right\}.$$

**Theorem 3.** Cubature formula

$$\Phi_1^3(m,n,p)=\int_0^1\int_0^1\int_0^1 J\,f(x,y,z)\sin 2\pi mx\sin 2\pi ny\sin 2\pi pz\,dxdydz \qquad (12)$$

has the following error of approach

$$\left|I_1^3(m,n,p)-\Phi_1^3(m,n,p)\right|\le\frac{\widetilde{M}}{64\ell^3}.$$

**2 case:** the input information about function is a set of traces of function on lines

Let

$$\tilde{J}_1 f(x,y,z)=\sum_{\tilde{k}=1}^{\ell^{3/2}} f(\tilde{x}_{\tilde{k}},y,z)\tilde{h}^0_{1\tilde{k}}(x)\ ,\quad \tilde{J}_2 f(x,y,z)=\sum_{\tilde{j}=1}^{\ell^{3/2}} f(x,\tilde{y}_{\tilde{j}},z)\tilde{h}^0_{2\tilde{j}}(y)\ ,$$

$$\tilde{J}_3 f(x,y,z)=\sum_{\tilde{s}=1}^{\ell^{3/2}} f(x,y,\tilde{z}_{\tilde{s}})\tilde{h}^0_{3\tilde{s}}(z)\,, \tag{13}$$

$$\tilde{h}^0_{1\tilde{k}}(x)=\begin{cases}1, x\in\tilde{X}_{\tilde{k}},\\ 0, x\notin\tilde{X}_{\tilde{k}},\end{cases}\quad \tilde{h}^0_{2\tilde{j}}(y)=\begin{cases}1, y\in\tilde{Y}_{\tilde{j}},\\ 0, y\notin\tilde{Y}_{\tilde{j}},\end{cases}\quad \tilde{h}^0_{3\tilde{s}}(z)=\begin{cases}1, z\in\tilde{Z}_{\tilde{s}},\\ 0, z\notin\tilde{Z}_{\tilde{s}},\end{cases},$$

$$\tilde{x}_{\tilde{k}}=\tilde{k}\Delta_1-\frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}}=\tilde{j}\Delta_1-\frac{\Delta_1}{2},\ \tilde{z}_{\tilde{s}}=\tilde{s}\Delta_1-\frac{\Delta_1}{2},\quad \Delta_1=\frac{1}{\ell^{3/2}},\quad \tilde{k},\tilde{j},\tilde{s}=\overline{1,\ell^{3/2}}\,.$$

**2 case:** the input information about function is a set of traces of function on lines

We define operator

$$\widetilde{J}f(x,y,z)=J_1\widetilde{J}_2f(x,y,z)+J_1\widetilde{J}_3f(x,y,z)-J_1\widetilde{J}_2\widetilde{J}_3f(x,y,z)+J_2\widetilde{J}_1f(x,y,z)+$$
$$+J_2\widetilde{J}_3f(x,y,z)-J_2\widetilde{J}_1\widetilde{J}_3f(x,y,z)+J_3\widetilde{J}_1f(x,y,z)+J_3\widetilde{J}_2f(x,y,z)-J_3\widetilde{J}_1\widetilde{J}_2f(x,y,z)- \quad (14)$$
$$-J_1J_2f(x,y,z)-J_1J_3f(x,y,z)-J_2J_3f(x,y,z)+J_1J_2J_3f(x,y,z).$$

on the following class of function

$$H_1^{3,1}\left(M,\overline{M},\widetilde{M}\right):\quad \left|f^{(1,0,0)}(x,y,z)\right|\le M,\quad \left|f^{(0,1,0)}(x,y,z)\right|\le M,\quad \left|f^{(0,0,1)}(x,y,z)\right|\le M,$$
$$\left|f^{(1,1,0)}(x,y,z)\right|\le \overline{M},\quad \left|f^{(1,0,1)}(x,y,z)\right|\le \overline{M},\quad \left|f^{(0,1,1)}(x,y,z)\right|\le \overline{M},\quad \left|f^{(1,1,1)}(x,y,z)\right|\le \widetilde{M}.$$

**2 case:** the input information about function is a set of traces of function on lines

**Theorem 4.** Cubature formula

$$\tilde{\Phi}_1^3(m,n,p)=\int_0^1\int_0^1\int_0^1 \tilde{J}f(x,y,z)\sin 2\pi mx\sin 2\pi ny\sin 2\pi pz\,dx\,dy\,dz \qquad (15)$$

has the following error of approach $\left|I_1^3(m,n,p)-\tilde{\Phi}_1^3(m,n,p)\right|\le\left(\frac{\tilde{M}}{64}+\frac{3\bar{M}}{16}\right)\frac{1}{\ell^3}.$

**3 case**: the input information about function is a set of values of the function in the points

Let
$$\overline{J}_1 f(x,y,z)=\sum_{\overline{k}=1}^{\ell^3} f(\overline{x}_{\overline{k}},y,z)\overline{h}_{1\overline{k}}^{0}(x),\quad \overline{J}_2 f(x,y,z)=\sum_{\overline{j}=1}^{\ell^3} f(x,\overline{y}_{\overline{j}},z)\overline{h}_{2\overline{j}}^{0}(y), \tag{16}$$

$$\overline{J}_3 f(x,y,z)=\sum_{\overline{s}=1}^{\ell^3} f(x,y,\overline{z}_{\overline{s}})\overline{h}_{3\overline{s}}^{0}(z),$$

$$\overline{h}_{1\overline{k}}^{0}(x)=\begin{cases}1, x\in \overline{X}_{\overline{k}},\\ 0, x\notin \overline{X}_{\overline{k}},\end{cases}\quad \overline{h}_{2\overline{j}}^{0}(y)=\begin{cases}1, y\in \overline{Y}_{\overline{j}},\\ 0, y\notin \overline{Y}_{\overline{j}},\end{cases}\quad \overline{h}_{3\overline{s}}^{0}(z)=\begin{cases}1, z\in \overline{Z}_{\overline{s}},\\ 0, z\notin \overline{Z}_{\overline{s}},\end{cases}$$

$$\overline{x}_{\overline{k}}=\overline{k}\Delta_2-\frac{\Delta_2}{2},\quad \overline{y}_{\overline{j}}=\overline{j}\Delta_2-\frac{\Delta_2}{2},\quad \overline{z}_{\overline{s}}=\overline{s}\Delta_2-\frac{\Delta_2}{2},\quad \Delta_2=\frac{1}{\ell^3},\quad \overline{k},\overline{j},\overline{s}=\overline{1,\ell^3}.$$

**3 case**: the input information about function is a set of of values of the function in the points

We define operator

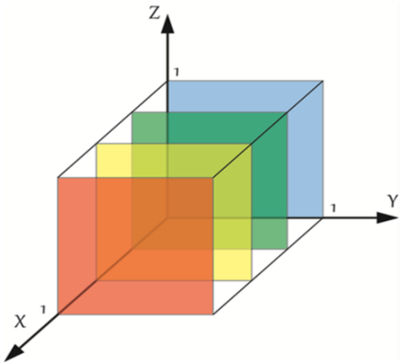
$$\bar{J}f(x,y,z)=J_1\tilde{J}_2\bar{J}_3f(x,y,z)+J_1\tilde{J}_3\bar{J}_2f(x,y,z)-J_1\tilde{J}_2\tilde{J}_3f(x,y,z)+$$
$$+J_2\tilde{J}_1\bar{J}_3f(x,y,z)+J_2\tilde{J}_3\bar{J}_1f(x,y,z)-J_2\tilde{J}_1\tilde{J}_3f(x,y,z)+ \qquad (17)$$
$$+J_3\tilde{J}_1\bar{J}_2f(x,y,z)+J_3\tilde{J}_2\bar{J}_1f(x,y,z)-J_3\tilde{J}_1\tilde{J}_2f(x,y,z)-$$
$$-J_1J_2\bar{J}_3f(x,y,z)-J_1J_3\bar{J}_2f(x,y,z)-J_2J_3\bar{J}_1f(x,y,z)+J_1J_2J_3f(x,y,z).$$

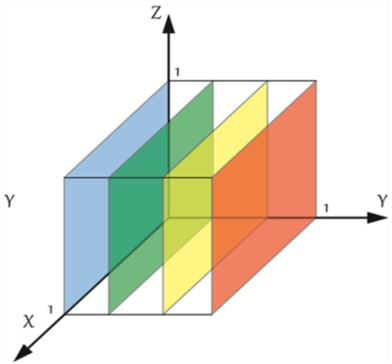
**Theorem 5.** Cubature formula

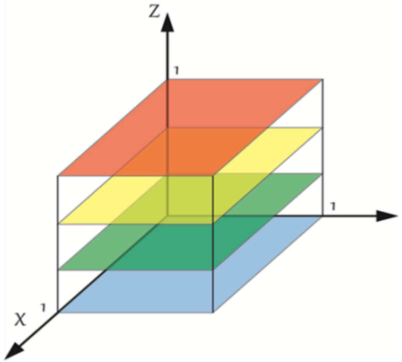
$$\bar{\Phi}_1^3(m,n,p)=\int_0^1\int_0^1\int_0^1 \bar{J}f(x,y,z)\sin 2\pi mx\sin 2\pi ny\sin 2\pi pz\,dxdydz \qquad (18)$$

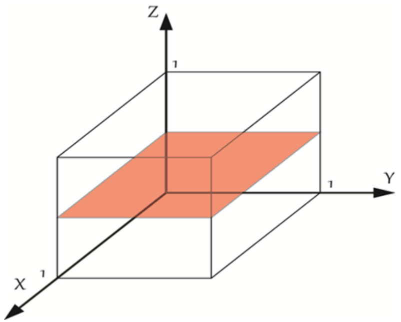
has the following error of approach $\left|I_1^3(m,n,p)-\bar{\Phi}_1^3(m,n,p)\right|\le\left(\frac{\tilde{M}}{64}+\frac{3\bar{M}}{16}+\frac{9}{4}M\right)\frac{1}{\ell^3}.$

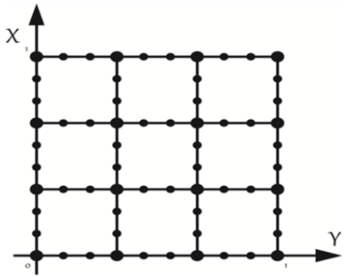
## The choice of knots



**Theorem 6.** Let

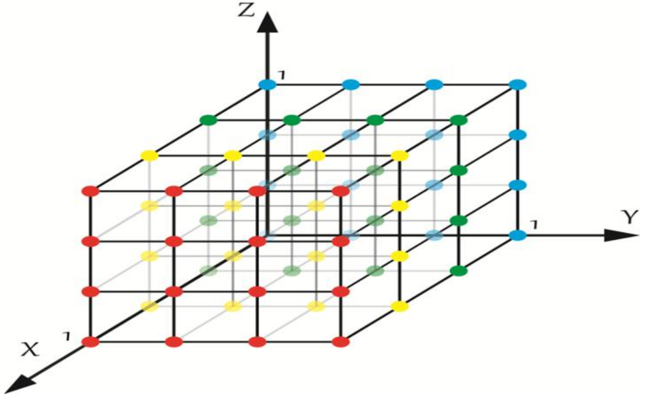
$$H_1^{3,1}(M) = \{ f(x,y,z) : \left| f^{(1,0,0)}(x,y,z) \right| \le M,\ \left| f^{(0,1,0)}(x,y,z) \right| \le M, \left| f^{(0,0,1)}(x,y,z) \right| \le M \},$$

then cubature formula

$$\widehat{\Phi}_1^3(m,n,p) = \sum_{\overline{k}=1}^{\ell^3} \sum_{\overline{j}=1}^{\ell^3} \sum_{\overline{s}=1}^{\ell^3} f\left(\overline{x}_{\overline{k}}, \overline{y}_{\overline{j}}, \overline{z}_{\overline{s}}\right) \int_{\overline{x}_{\overline{k}-\frac{1}{2}}}^{\overline{x}_{\overline{k}+\frac{1}{2}}} \sin 2\pi mx dx \int_{\overline{y}_{\overline{j}-\frac{1}{2}}}^{\overline{y}_{\overline{j}+\frac{1}{2}}} \sin 2\pi ny dy \int_{\overline{z}_{\overline{s}-\frac{1}{2}}}^{\overline{z}_{\overline{s}+\frac{1}{2}}} \sin 2\pi pz dz$$

has the following error of approach:

$$\left| I_1^3(m,n,p) - \widehat{\Phi}_1^3(m,n,p) \right| \le \frac{3}{4} M \frac{1}{\ell^3}.$$



## Computer experiment

Calculation of $I_1^3(25,25,25)$ for $f(x,y,z)=\sin(x+y+z)$, $\ell=4$ by proposed and classic formulas:

| | $\bar{\Phi}_1^3(m,n,p)$ | $\hat{\Phi}_1^3(m,n,p)$ |
|---|---|---|
| E | $3{,}1\cdot 10^{-11}$ | $1{,}0\cdot 10^{-13}$ |
| Q | 2048 | 262144 |
| M | 78209932 | 78594620 |
| T | 2,1 | 68,7 |

E – calculation accuracy, Q – amount of knots, M – storage capacity, T – time of calculation.

## TWO DIMENSIONAL CASE


- O.N. Lytvyn, O.P. Nechuyviter, Formulas for an Evaluation of Fourier Coeffitients of Functions of Two Variables with using Spline-interlineation. Reports of National Academy of Science of Ukraine. – Kiev. –1998. – № 1. – P. 23-28.
- O.N. Lytvyn, O.P. Nechuyviter, An Optimal by the Order of Exactness Cubature Formula of Calculation of Highly Oscillatory Integrals of Functions Two Variables with using Spline-interlineation. Reports of National Academy of Science of Ukraine. – Kiev. – 2006. – № 6. – P. 9-13.
- O.N. Lytvyn, O.P. Nechuiviter, About One Cubature Formula for Calculation 2D – Fourier Coefficients with the use of Interlineation. Reports of National Academy of Science of Ukraine. – Kiev. –2010. – № 3. – P. 24-29.
- O.N. Lytvyn, O.P. Nechuyviter, Methods in the Multivariate Digital Signal Processing with Using Spline-interlineation. Proceeding of IASTED International Conferences on Automation, Control and Information Technology (ASIT 2010) (June 15 – 18, 2010). – Novosibirsk. – 2010. – P. 90 – 96.

## THREE DIMENSIONAL CASE


- O. M. Lytvyn, Y. I. Pershina, O.P. Nechuyviter, O.O. Litvin, S.I. Kuyik Construction Methods of 4D Mathematical Models 3D Bodies on a Basis Interlineation, Interflatation, Blending Approximation and Wavelets. *What Where When Multi-dimensional Advances for Industrial Process Monitoring: Poceedings of International Symposium (23-24 June 2009). – Leeds, UK. – 2009. – P. 443- 453.*

- O. N. Lytvyn, O. P. Nechuyviter, 3D Fourier Coefficients on the Class of Differentiable Functions and Spline Interflatation. *Journal of Automation and Information Sciences. – 2012. – Vol. 44, № 3. – pp. 45–56.*

- *O.N. Lytvyn, O.P. Nechuiviter, 3D - Fourier Coefficients on the Class of Differential Functions and Spline-interflatation. Reports of National Academy of Science of Ukraine. – Kiev. –2012. – № 3. – P. 45-50.*

# THREE DIMENSIONAL CASE


- O.N. Lytvyn, O.P. Nechuiviter, About One Cubature Formula for Calculation 2 D – Fourier Coefficients with the Use of Interlineation. Reports of *National Academy of Science of Ukraine.* – Kiev. –2012. – № 8. – P. 36-41.

- O. N. Lytvyn, O. P. Nechuiviter, Approximate Calculation of Integrals of Highly Oscillatory Functions of Three Variables with the Use of Interflatation. *Announcer of Moscow State Regional University, Russia. Series: Physics and Mathematics. – 2013. – № 2. – С. 3–9.*

- O. N. Lytvyn, O. P. Nechuiviter, Approximate Calculation of Triple Integrals of Rapidly Oscillating Functions with the Use of Lagrange Polynomial Interflatation. *Cybernetics and Systems Analysis. – 2014. – Vol. 50, № 3. – pp. 410–418.*

## TWO AND THREE DIMENSIONAL IRREGULAR CASE


- Iserles A., On the numerical quadrature of highly oscillating integrals II: Irregular oscillators / A. Iserles // IMA J. Numer. Anal. – 2005. – № 25. – P. 25–44.
- J. *Gao, M. Condon, A. Iserles*, Spectral computation of highly oscillatory integral equations in laser theory. Tech. Re-ports Numerical Analysis (NA2018/04) DAMPT: University of Cambridge, 2018.

- V.I. Mezhuyev, O.M. Lytvyn, O.P. Nechuiviter, Y.I. Pershyna, O.O. Lytvyn, K.V. Keita, Cubature formula for approximate calculation of integrals of two-dimensional irregular highly oscillating functions, *U.P.B. Sci. Bull., Series A*, vol. 80(3), pp. 169–182, 2017.
- O.P. Nechuiviter, Cubature formula for approximate calculation integral of highly oscillating function of tree variables (irregular case), *Radio Electronics, Computer Science, Control*, vol. 4, pp. 65–73, 2020. doi: 10.15588/1607-3274-2020-4-7.

A two-dimensional integral from highly oscillating functions of general view is defined as

$$I^2(\omega) = \int_0^1 \int_0^1 f(x,y)\, e^{i\omega g(x,y)}\, dxdy$$

for $f(x,y),\ g(x,y) \in H^{2,1}(M,\tilde{M})$.

We are looking for two operators: the first

$$Jf(x,y) = \sum_{k=1}^{\ell_1} f(x_k,y)\, h1_{0k}(x) + \sum_{j=1}^{\ell_1} f(x,y_j)\, H1_{0j}(y) - \sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1} f(x_k,y_j)\, h1_{0k}(x)\, H1_{0j}(y)$$

and the second one

$$Og(x,y) = \sum_{p=1}^{\ell_2} f(x_p,y)\, h2_{0p}(x) + \sum_{s=1}^{\ell_2} f(x,y_s)\, H2_{0s}(y) - \sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2} f(x_p,y_s)\, h2_{0p}(x)\, H2_{0s}(y).$$

# TWO DIMENSIONAL IRREGULAR CASE

$$h1_{0k}(x)=\begin{cases}1, x\in X1_k,\\0, x\notin X1_k,\end{cases}\quad k=\overline{1,\ell_1},\qquad H1_{0j}(y)=\begin{cases}1, y\in Y1_j,\\0, y\notin Y1_j,\end{cases}\quad j=\overline{1,\ell_1},$$

$$X1_k=\left[x_{k-1/2}, x_{k+1/2}\right],\ Y1_j=\left[y_{j-1/2}, y_{j+1/2}\right],$$

$$x_k=k\Delta_1-\Delta_1/2,\qquad y_j=j\Delta_1-\Delta_1/2,\ k,j=\overline{1,\ell_1},\ \Delta_1=1/\ell_1,$$

$$h2_{0p}(x)=\begin{cases}1, x\in X2_p,\\0, x\notin X2_p,\end{cases}\quad p=\overline{1,\ell_1},\qquad H2_{0j}(y)=\begin{cases}1, y\in Y2_s,\\0, y\notin Y2_s,\end{cases}\quad s=\overline{1,\ell_1},$$

$$X1_p=\left[x_{p-1/2}, x_{p+1/2}\right],\ Y1_s=\left[y_{s-1/2}, y_{s+1/2}\right],$$

$$x_p=p\Delta_2-\Delta_2/2,\qquad y_s=s\Delta_2-\Delta_2/2,\ \text{p, s}=\overline{1,\ell_2},\ \Delta_2=1/\ell_2.$$

The following cubature formula $\Phi^2(\omega)=\int_0^1\int_0^1 Jf(x,y)\,e^{i\omega Og(x,y)}\,dxdy$

is suggesting for numerical calculation of $I^2(\omega)=\int_0^1\int_0^1 f(x,y)\,e^{i\omega g(x,y)}\,dxdy$ .

**Theorem 6.** Suppose that $f(x,y),\ g(x,y)\in H^{2,1}\left(M,\tilde{M}\right)$. Let functions $f(x,y),\ g(x,y)$ be defined by $N=2\ell_1+2\ell_2$ traces $f\left(x_k,y\right),\ k=\overline{1,\ell_1}$, $f\left(x,y_j\right),\ j=\overline{1,\ell_1}$ and $g(x_p,y),\ p=\overline{1,\ell_2}$, $g(x,y_s),\ s=\overline{1,\ell_2}$ on the systems of perpendicular lines in domain $G=[0,1]^2$. It is true that

$$\rho\left(I^2(\omega),\Phi^2(\omega)\right)=$$

$$=\left|\int_0^1\int_0^1 f(x,y)\,e^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1 Jf(x,y)\,e^{i\omega Og(x,y)}\,dxdy\right|\le\frac{\tilde{M}}{16}\frac{1}{\ell_1^2}+\tilde{M}\min\left(2;\frac{\tilde{M}\omega}{16}\frac{1}{\ell_2^2}\right).$$

# THREE DIMENSIONAL IRREGULAR CASE

It is necessary to construct and investigate the cubature formula of the approximate calculation of the integral of highly oscillating function in a general case

$$I^3(\omega) = \int_0^1 \int_0^1 \int_0^1 f(x,y,z) e^{i\omega g(x,y,z)} dxdydz,$$

when the following information is given by traces on the planes.

The following cubature formula

$$\Phi^3(\omega) = \int_0^1 \int_0^1 \int_0^1 Jf(x,y,z) e^{i\omega Og(x,y,z)} dxdydz$$

is proposed for numerical calculation of integral $I^3(\omega)$.

**Theorem 7.** Suppose that $f(x,y,z) \in H^{3,1}\left(M,\widetilde{M}\right)$, $g(x,y,z) \in H^{3,1}\left(M,\widetilde{M}\right)$. Let functions $f(x,y,z)$, $g(x,y,z)$ be defined by $N = 3\ell_1 + 3\ell_2$ traces $f(x_k,y,z)$, $f(x,y_j,z)$, $f(x,y,z_s)$, $k,j,s = \overline{1,\ell_1}$ and $g(\tilde{x}_p,y,z)$, $g(x,\tilde{y}_q,z)$, $g(x,y,\tilde{z}_r)$, $p,q,r = \overline{1,\ell_2}$ on the systems of perpendicular planes in domain $G = [0,1]^3$. It is true that

$$\rho\left(I^3(\omega), \Phi^3(\omega)\right) \le \frac{\widetilde{M}}{64}\frac{1}{\ell_1{}^3} + \widetilde{M}\min\left(2; \frac{\widetilde{M}\omega}{64}\frac{1}{\ell_2{}^3}\right).$$

# THREE DIMENSIONAL IRREGULAR CASE

Information about functions $f(x,y,z)$ and $g(x,y,z)$ is given by the corresponding traces of functions on mutually perpendicular planes. The Table 3 shows the results of calculations $I^3(\omega)$ for $f(x,y,z)=\sin(x+y+z)$ and $g(x,y,z)=\cos(x+y+z)$ for different $\ell_1, \ell_2$ and for $\omega=2\pi, \omega=5\pi, \omega=10\pi$. For each case, table 3 shows the value of the obtained approximation error

$$\varepsilon = \left| I^3(\omega) - \Phi^3(\omega) \right|,$$

$$E = \frac{\widetilde{M}}{64}\frac{1}{\ell_1{}^3} + \widetilde{M}\min\left(2; \frac{\widetilde{M}\omega}{64}\frac{1}{\ell_2{}^3}\right).$$

# THREE DIMENSIONAL IRREGULAR CASE

Table 3 – Calculation $I^3(\omega)$ by cubature formula $\Phi^3(\omega)$

| $\ell$ | $\omega$ | $\mathrm{Re}\left(\Phi^3(\omega)\right)$ | $\mathrm{Im}\left(\Phi^3(\omega)\right)$ | $\varepsilon$ | $E$ |
|---|---|---|---|---|---|
| 5 | $10\pi$ | -0,00180433697415137 | 0,000356265110351913 | $7,39\cdot10^{-4}$ | $4,05\cdot10^{-3}$ |
| 10 | $10\pi$ | -0,00140102828305083 | -0,000261065518418426 | $3,62\cdot10^{-6}$ | $5,06\cdot10^{-4}$ |
| 15 | $10\pi$ | -0,00139749596727245 | -0,000261837407624295 | $2,41\cdot10^{-7}$ | $1,05\cdot10^{-4}$ |
| 20 | $10\pi$ | -0,00035388799218903 | -0,000260805717278864 | $1,39\cdot10^{-6}$ | $6,33\cdot10^{-5}$ |
| 25 | $10\pi$ | -0,00139663624810749 | -0,00026136986950217 | $8,36\cdot10^{-7}$ | $3,24\cdot10^{-5}$ |

Obviously, functions $\sin(x+y+z)$, $\cos(x+y+z)$ are the functions that belong to a broader class of functions. This suggests that the cubature formula $\Phi^3(\omega)$ has a good approximation accuracy and should be researched on other classes of functions.

# Conclusions

- Report is dedicated to the constructing cubature formulas of approximate calculation of integrals of highly oscillatory functions of two and three variables.

- Cubature formulas are based on Failon method and special operators.

- The feature of the proposed cubature formulas is using the input information about function as a
  - set of traces of function on planes;
  - set of traces of function on lines;
  - set of values of the function in the points.

# Thank you for attention!