Exact spectral asymptotics of fractional processes and its applications

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Modeling roughness and long-range dependence with fractional processes
8th European Congress of Mathematics

Outline

Outline

Objects of study

We consider

- Zero mean Gaussian process $X = (X_t; t \in [0,1])$ with covariance function

$$K(s,t) = \mathbb{E} X_s X_t, \quad s,t \in [0,1]$$

- Covariance operator

$$f \mapsto (Kf)(t) := \int_0^1 K(s,t) f(s) ds, \quad t \in [0,1]$$

A spectral problem

Problem

Given a covariance operator K, compute its eigenvalues and eigenfunctions, i.e. solve the equation:

$$(K\varphi_n)(t) = \lambda_n \varphi_n(t), \quad t \in [0, 1].$$

Unfortunately λ_n and φ_n are rarely available in closed form

Outline

Applications

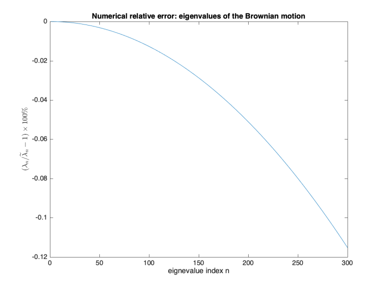
- Karhunen-Loeve expansion
- Equivalence and orthogonality of Gaussian measures
- Approximate sampling from heavy tailed distributions
- Numerical solutions of stochastic equations
- **Exact asymptotics of the small ball probabilities**
- **Asymptotics of solutions of integral equations**
- ...

Some comments: numerical methods

Numerical approximations. . .

- . . . give reliable results for several *first* eigenelements
- . . . **do not work** for **large** values of **n**

Numerical relative error for eigenvalues of Brownian motion



Small L_2-ball probabilities

Problem

Given a process $X = (X_t, t \in [0,1])$, find the asymptotics of

$$\mathbb{P}(\|X\| \le \varepsilon) \quad \text{as } \varepsilon \to 0.$$

To solve it, we need asymptotics

$$\lambda_n = \sum_{j=1}^{k} c_j n^{-d_j} + O(n^{-\gamma}),$$

where $\gamma - d_1 > 1, 0 \le d_j - d_1 \le 1$.

Why it is non trivial

Discontinuity of the second order term

Let $K(s,t) = s \wedge t - \varepsilon s\, t \quad \varepsilon \in [0,1]$. Then

$$\lambda_n = \nu_n^{-2}$$

where

- $\nu_n = n\pi - \frac{1}{2}\pi + O(n^{-1})$ when $\varepsilon \in [0,1)$
- $\nu_n = n\pi$ when $\varepsilon = 1$.

Solutions of second kind integral equations

Two variants of the problem

- $(\varepsilon \to 0)$ Singularly perturbed integral equations

$$\varepsilon u_\varepsilon(x) + (K u_\varepsilon)(x) = f(x), \quad x \in [0, 1]$$

- $(T \to \infty)$ Large time behaviour of the solution

$$u_T(x) + (K u_T)(x) = f(x), \quad x \in [0, T]$$

They arise in

- optimal linear filtering/interpolation problems
- statistical inference of processes

Singular perturbations

Fix $\varepsilon > 0$ and let g_ε be the solution of the equation:

$$\varepsilon g_\varepsilon(u) + \frac{d}{du}\int_0^1 g_\varepsilon(v)|u-v|^{2H-1} sign(u-v)dv = 1,\ u \in [0,1],$$

An important question

What can we say about g_ε when $\varepsilon \to 0$?

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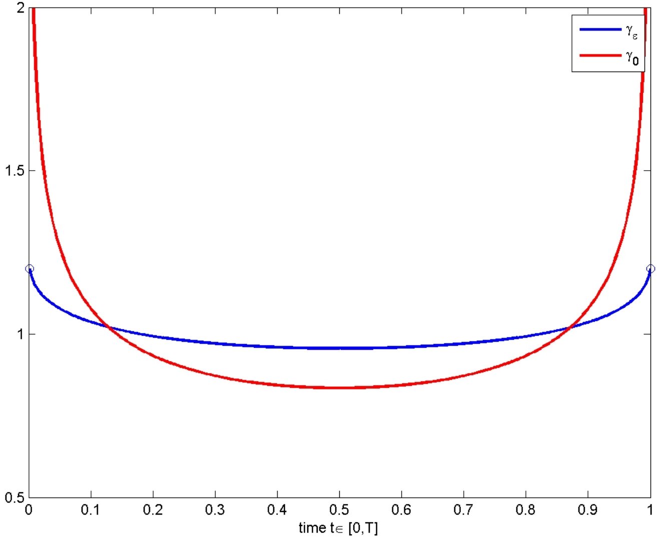
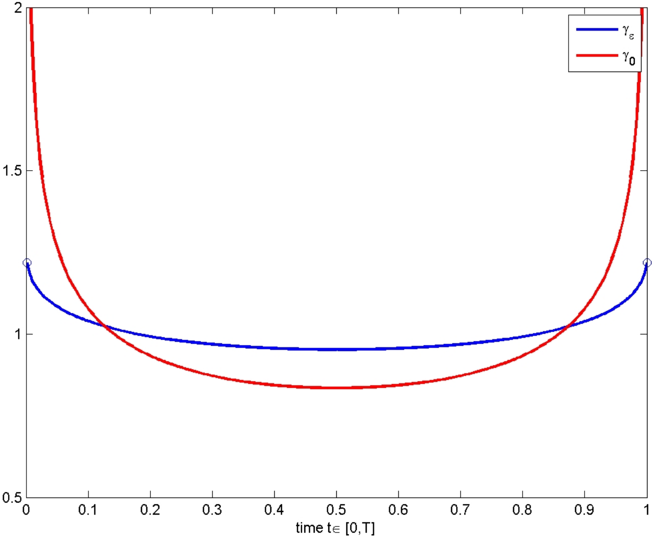
What can we say about g_ε when $\varepsilon \to 0$?

g_ε when $\varepsilon \to 0$



g_ε when $\varepsilon \to 0$



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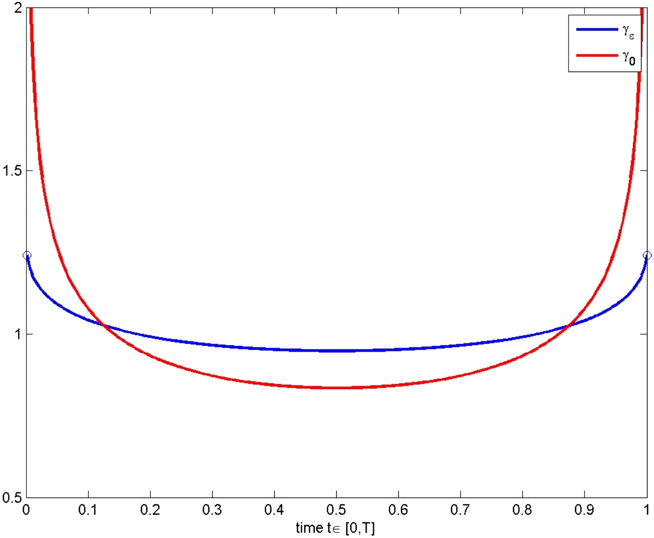

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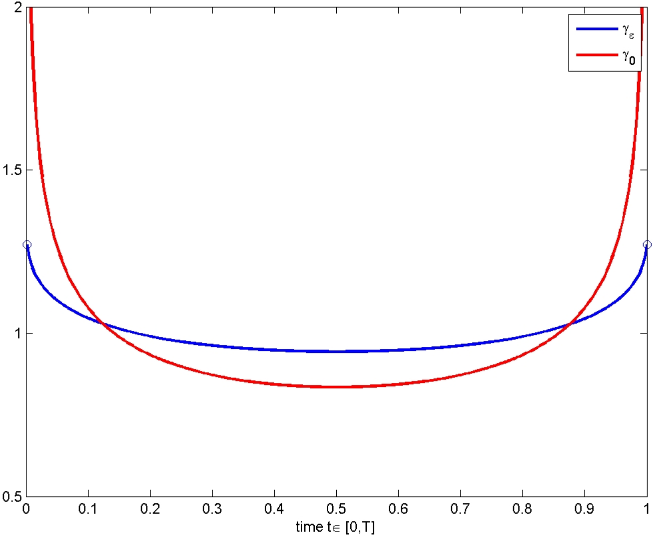

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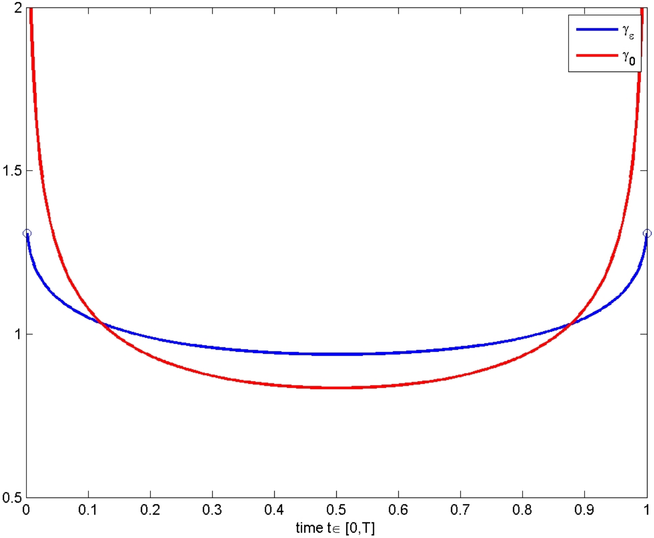

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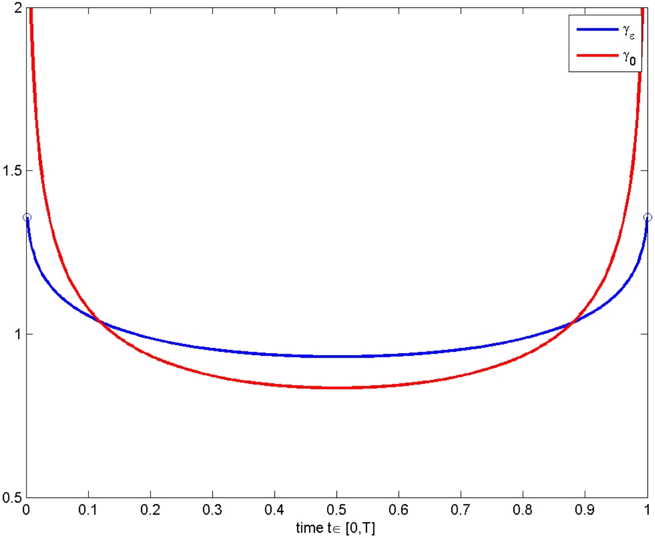

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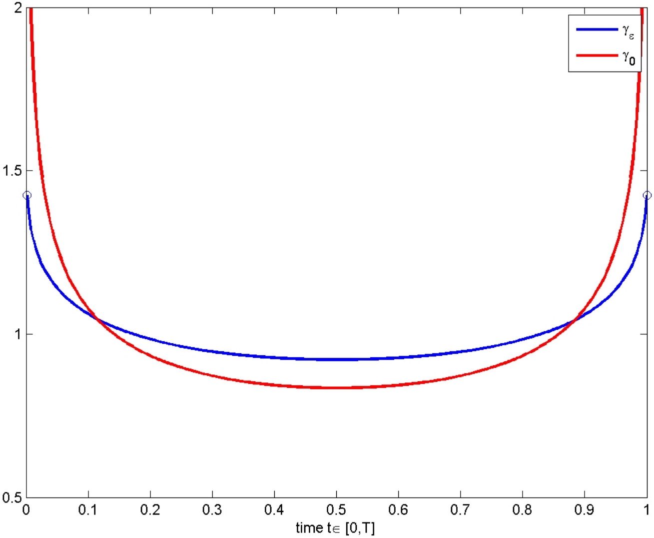

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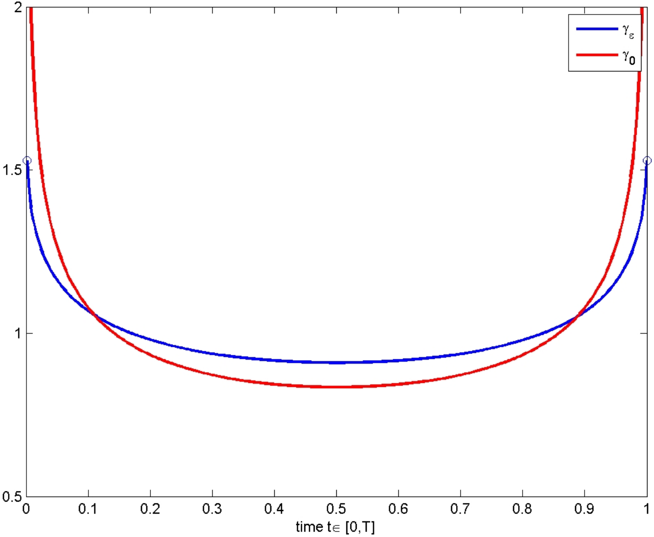

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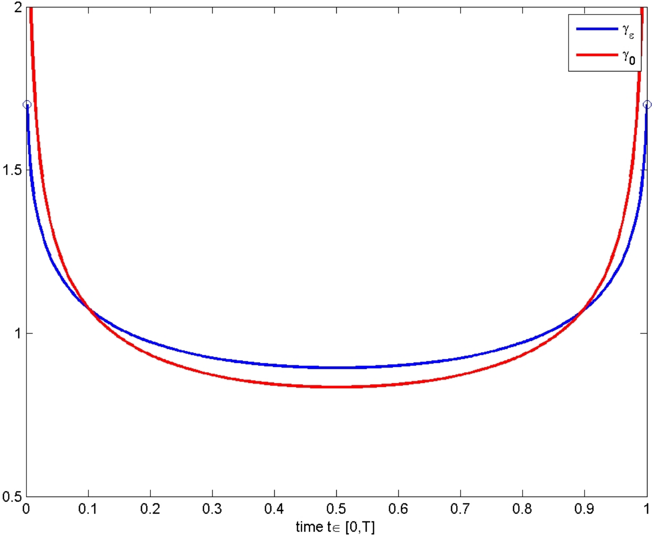

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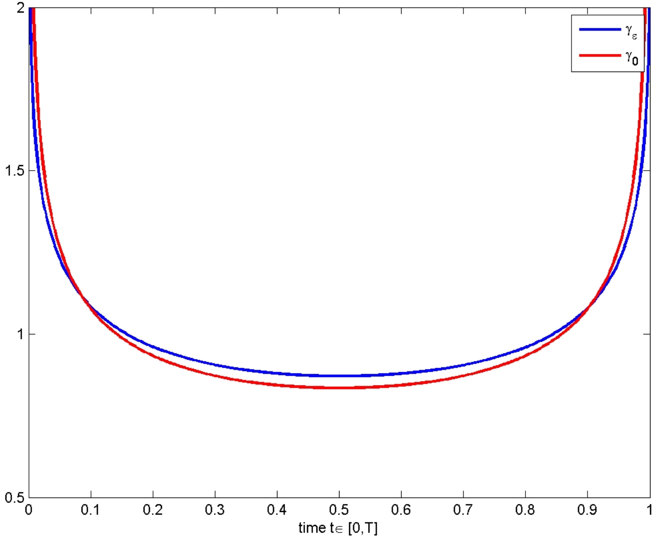

Outline

State-of-the-art

Asymptotics for Brownian motion and related processes

- For the Brownian motion $K(s,t) = \min(s,t)$

$$\lambda_n = \frac{1}{(n-\frac{1}{2})^2\pi^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2}\sin\left(n-\tfrac{1}{2}\right)\pi t$$

- Similar results for related processes (Brownian bridge, Ornstein–Uhlenbeck process, etc.)

State-of-the-art techniques

- Reduce the original eigenproblem to a **classical** Sturm-Liouville problem.
- This does not work for a "long memory" processes

State-of-the-art

Asymptotics for fractional Brownian motion

- For the fractional Brownian motion the leading asymptotic term

$$\lambda_n = \frac{\sin(\pi H)\Gamma(2H+1)}{(n\pi)^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5}+\delta}\right)$$

  Works: J. C. Bronski (2003), H. Luschgy and G. Pages (2004), A. Nazarov and Ya. Nikitin (2004)

- Nothing was known about the **eigenfunctions**.

Classical and Fractional Sturm-Liouville problem

Brownian motion (well-known)	Fractional Liouville (new)
Spectral problem	Spectral problem
$\Uparrow$	$\Downarrow$
Sturm-Liouville Problem	Sturm-Liouville Problem

Details for Brownian Motion

Brownian motion W_t

Spectral problem:

$$\int_0^1 x \wedge y \phi(y)\, ds = \lambda \phi(x), \quad x \in [0,1].$$

Sturm-Liouville problem (classical):

$$-\phi'' = \lambda^{-1}\phi, \quad x \in [0,1]$$

$$\phi(0) = \phi'(1) = 0$$

Solution to spectral problem:

$$\lambda_n = \frac{1}{(n-\frac{1}{2})^2\pi^2} \quad \text{and} \quad \phi_n(t) = \sqrt{2}\sin\left(\left(n-\tfrac{1}{2}\right)\pi t\right)$$

Details for Riemann-Liouville process

Riemann-Liouville process $\int_0^t (t-s)^{\alpha-1} dW_s$

Sturm-Liouville problem (fractional):

$$
\begin{aligned}
&{}^cD^\alpha_{1-}\, {}^cD^\alpha_{0+} u(x) = \lambda^{-1} u(x), \quad x \in [0,1], \\
&u(0) = 0, \ {}^cD^\alpha_{0+} u(1) = 0,
\end{aligned}
$$

with the left and right Caputo derivatives of order $\alpha \in (0,1)$.

Spectral problem:

$$
\int_0^1 K(x,y) f(y) dy = \lambda f(x), \quad x \in [0,1],
$$

with the kernel

$$
K(x,y) = \frac{1}{\Gamma(\alpha)^2} \int_0^{x\wedge y} (x-y)^{\alpha-1} (y-t)^{\alpha-1} dt,
$$

Outline

Essence of our work

Problem solved

Obtain asymptotics of

- eigenfunctions with respect to the uniform norm
- eigenvalues up to the second/third. . . order terms

for a large class of fractional processes.

Main tool

- Spectral problem reduced to an equivalent integro-algebraic system of equations, more amenable to asymptotic analysis
- The method is a transposition of techniques for Riemann boundary value problems

inspired by K. Case (1960), A. Gibbs (1969), S. Ukai (1970)

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Class C of operators

The method, in principle, applies to

- operators with difference kernels representable as

$$K(u) = \int_0^\infty \kappa(t) e^{-t|u|} dt,$$

  i.e. having inverse Laplace transform.
- their compositions with the integration operator
- linear combinations (mixed Gaussian processes)

But the implementation of the method

- is very specific to the fine structure of the kernel
- requires different tricks and leads to entirely unexpected outcomes

Operators of class C, I

fBm "derivative"

$$(Kf)(t) = \frac{d}{dt}\int_0^1 f(y)H|t-y|^{2H-1}\,\mathrm{sign}(t-y)dy.$$

fBm

$$(Kf)(t) = \int_0^1 K(t,y)f(y)dy,$$

where

$$K(x,y) = \frac{1}{2}\left(x^{2H} + y^{2H} - |x-y|^{2H}\right), \quad x, y \in [0,1].$$

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Operators of class C, II

fBm integral

$$K(x,y)=\int_0^t\int_0^s \tfrac{1}{2}\left(u^{2H}+v^{2H}-|v-u|^{2H}\right)dudv$$

Fractional Ornstein–Uhlenbeck

$$K(x,y)=\int_0^t e^{\beta(t-v)}\frac{d}{dv}\int_0^s H|v-u|^{2H-1}\,\mathrm{sign}(v-u)e^{\beta(s-u)}dudv.$$

Operators of class C, II

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Operators of class C, III

Mixed fBm

Mixture with an independent standard Brownian motion B
$X_t = B_t + B_t^H, \quad t \in [0,1],$

$$K(x,y) = s \wedge t + c_\alpha \int_0^t \int_0^s |u-v|^{-\alpha} du dv.$$

with $\alpha := 2 - 2H \in (0,1)$ and $c_\alpha := (1 - \frac{\alpha}{2})(1-\alpha)$

Outline

"Derivative" of the fBm

Eigenvalues

$$\lambda_n = \sin(\pi H)\Gamma(2H+1)\nu_n^{1-2H}, \quad n = 1, 2, \ldots, \text{where}$$

$$\nu_n = \left(n - \frac{1}{2}\right)\pi + \frac{1-2H}{4}\pi + O(n^{-1}) \quad \text{as } n \to \infty.$$

Eigenfunctions

$$\varphi_n(x) = \sqrt{2}\cos\left(2\nu_n x - \frac{1+\alpha}{8}\pi\right) + \sqrt{|\alpha-1|}\frac{1}{\pi}\int_0^\infty |\rho_0(\tau)|\left(e^{-2\nu_n x\tau} - (-1)^n e^{-2\nu_n(1-x)\tau}\right)d\tau + n^{-1}r_n(x),\ x \in [0,1],\ \text{with } \alpha = 2 - 2H \in (0,2)\setminus\{1\}$$

fBm and fOU: Eigenvalues Asymptotics

Eigenvalues

The eigenvalues satisfy

$$\lambda_n = \sin(\pi H)\Gamma(2H+1)\frac{\nu_n^{1-2H}}{\nu_n^2+\beta^2}, \quad n = 1, 2, \dots$$

where ν_n is a sequence with the following asymptotics

$$\nu_n = (n-\frac{1}{2})\pi - \frac{(H-1/2)^2}{(H+1/2)}\frac{\pi}{2} + O_\beta(n^{-1}) \quad \text{as } n \to \infty.$$

fBm and fOU: Eigenfunctions Asymptotics

Eigenfunctions

The corresponding normalized eigenfunctions admit the approximation

$$\varphi_n(x) = \sqrt{2}\sin\left(\nu_n x + \frac{H-3/2}{4}\frac{H-1/2}{H+1/2}\pi\right) + \int_0^\infty f_0(u)e^{-x\nu_n u}du + (-1)^n \int_0^\infty f_1(u)e^{-(1-x)\nu_n u}du + \nu_n^{-1} r_n(x),$$

where the residual $r_n(x)$ is bounded by a constant, depending only on H, and $f_j(u)$ is an explicit function.

A typical shape of the eigenfunctions

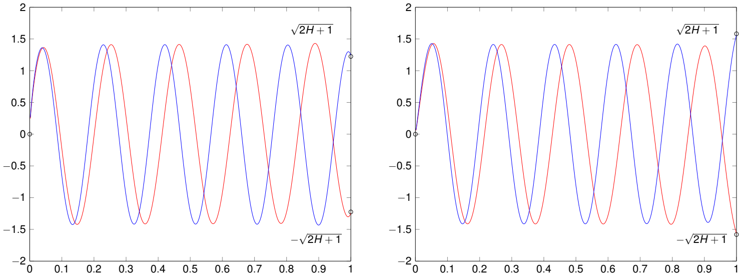

Figure: φ_{10} and φ_{11} for $H = \frac{1}{4}$ (left) and $H = \frac{3}{4}$ (right)

Integrated fBm Eigenvalues Asymptotics

Eigenvalues

The eigenvalues of covariance operator of integrated fBm satisfy

$$\lambda_n = \sin(\pi H)\Gamma(2H+1)\nu_n^{-2H-3} \quad n = 1, 2, \ldots$$

where

$$\nu_n = (n - \frac{1}{2})\pi - \frac{(H-1/2)(H+1/2)}{H+3/2}\frac{\pi}{2} + O(n^{-1}).$$

Integrated fBm Eigenfunctions Asymptotics

The corresponding eigenfunctions admit the approximation

$$\varphi_n(x) = \underbrace{\varphi_n^{(1)}(x)}_{\text{oscilatory term}} + \underbrace{\varphi_n^{(2)}(x)}_{\text{polynomial boundary layer}} + \underbrace{\varphi_n^{(3)}(x)}_{\text{exponential boundary layer}} + \underbrace{n^{-1} r_n(x)}_{\text{residual}}$$

where $r_n(x)$ is bounded uniformly in both $n \in \mathbb{N}$ and $x \in [0,1]$ and

$$\varphi_n^{(1)}(x) = \sqrt{2}\cos\left(\nu_n x + \frac{2H+1}{8}\pi - \frac{H-1/2}{H+3/2}\pi\right)$$

$$\varphi_n^{(2)}(x) = \int_0^\infty \rho_0(t)\left(Q_0(t)e^{-t\nu_n x} - (-1)^n Q_1(t)e^{-t\nu_n(1-x)}\right)dt$$

$$\varphi_n^{(3)}(x) = C_6 e^{-c\nu_n x}\cos\left(s\nu_n x + \varkappa_0\right) + C_7 e^{-c\nu_n(1-x)}\cos\left(s\nu_n(1-x) + \varkappa_1\right)$$

Mixed fBm spectral problem

Eigenvalues:

$$\lambda_n = \frac{1}{\nu_n^2} + \frac{\sin(\pi H)\Gamma(2H+1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \dots$$

where

$$\nu_n = \nu_n^{BM}\mathbf{1}_{\{H>\frac{1}{2}\}} + \nu_n^{fBM}\mathbf{1}_{\{H\le\frac{1}{2}\}} + O\big(n^{-|2H-1|}\big).$$

Eigenfunctions:

$$\varphi_n(x) = \varphi_n^{BM}(x)\mathbf{1}_{\{H>\frac{1}{2}\}} + \varphi_n^{fBm}(x)\mathbf{1}_{\{H\le\frac{1}{2}\}} + \frac{1}{n^{|2H-1|}} r_n(x)$$

Outline

Integro-differential Equation

Equation

$$\varepsilon g_\varepsilon(u)+\frac{d}{du}\int_0^1 g_\varepsilon(v)|u-v|^{2H-1} sign(u-v)dv = 1,\ u\in[0,1],$$

Corollary from our asymptotics

Convergence results for solutions:

- weak convergence with rate ε
- L^2 convergence with a rate depending on H
- boundary layer construction with $\frac{1}{\sqrt{\varepsilon}}$ rate

Exact Small Bals Probability, fBm

For all $H \in (0,1)$,

$$\mathbb{P}(\|B^H\|_2 \le \varepsilon) \simeq \varepsilon^{\gamma(H)} \exp\left(-\beta(H)\varepsilon^{-\frac{1}{H}}\right), \quad \varepsilon \to 0$$

where

$$\beta(H) = \frac{H}{(2H+1)^{\frac{2H+1}{2H}}} \left(\frac{\sin(\pi H)\Gamma(2H+1)}{\left(\sin\left(\frac{\pi}{2H+1}\right)\right)^{2H+1}}\right)^{\frac{1}{2H}}$$

and

$$\gamma(H) = \frac{1}{2H}\left(\frac{3}{4} + H^2 - H + \frac{1}{2}\right),$$

Exact Small Bals Probability, mixed fBm

$$\mathbb{P}\left(\|\widetilde{B}\|_2 \le \varepsilon\right) \sim \varepsilon^{\gamma} \exp\left(- \sum_{k=0}^{\left\lfloor \frac{1}{|2H-1|} \right\rfloor} \beta_k \varepsilon^{\frac{1}{H \wedge \frac{1}{2}} \left(k|2H-1|-1\right)} \right)$$

where γ and β_k, $k = 0, 1, 2, \ldots$ are explicitly defined functions of H.

Linear filtering of fBm

Large scale asymptotics, $T \to \infty$

$$Y_t = \mu \int_0^t B_s^H ds + B_t, \quad t \in [0, T]$$

The steady state filtering error is given by

$$\lim_{T\to\infty} P_T = \frac{\big(\sin(\pi H)\Gamma(2H+1)\big)^{\frac{1}{2H+1}}}{\sin \frac{\pi}{2H+1}} \mu^{-\frac{4H}{2H+1}}$$

Linear filtering of fOU

Small noise asymptotics, $\varepsilon \to 0$

$$Y_t = \mu \int_0^t X_s ds + \sqrt{\varepsilon} B_t, \quad t \in [0, T]$$

The *high signal-to-noise* filtering error is given by

$$P_\varepsilon(t) \simeq (\varepsilon/\mu^2)^{\frac{2H}{1+2H}} \frac{\Big(\sin(\pi H)\Gamma(2H+1)\Big)^{\frac{1}{1+2H}}}{\sin \frac{\pi}{2H+1}} \begin{cases} \frac{1}{2H+1} & t < T \\ 1 & t = T \end{cases}$$

Approach in a nutshell, I

For the Laplace transform (a priori analytic function!)

$$\hat{\varphi}(z) := \int_0^1 \varphi(x) e^{-zx} dx, \quad z \in \mathbb{C}$$

find an expression with handy singularities.

Approach in a nutshell, II

Reduce the original spectral problem to the solution of **Riemann-Hilbert boundary value problem** of finding two analytical on the cut plane $\mathbb{C}\setminus\mathbb{R}_{>0}$ functions $\mathbf{\Phi_0(z)}$ and $\mathbf{\Phi_1(z)}$ that satisfy

- *boundary condition* on $\mathbb{R}_{>0}$
- *a priori estimates* at $z = 0$ and *polynomial growth* rate at $z \to \infty$
- certain *algebraic conditions* on the imaginary axis.

Sketch of proof

- Using the particular structure of the eigenproblem obtain

$$\widehat{\varphi}(z) = P(z) - \frac{Q(z)}{\Lambda(z)}\Big(e^{-z}\Phi_1(-z) + \Phi_0(z)\Big), \quad z \in \mathbb{C}$$

  where

  - Φ_0 and Φ_1 are analytic on $\mathbb{C} \setminus \mathbb{R}_+$
  - $\Lambda(z)$ is an explicit function
  - $P(z)$ and $Q(z)$ are polynomials of a finite degree

- A calculation reveals that the function $\Lambda(z)$ has
  - has a finite number of zeros $z_1(\lambda), \dots, z_k(\lambda)$
  - jump discontinuity along real line

Sketch of proof

- Removal of the discontinuity gives conditions on the limit values

  $$\Phi_0^{\pm}(t) = \lim_{z\to t^{\pm}} \Phi_0^{\pm}(z) \quad \text{and} \quad \Phi_1^{\pm}(t) = \lim_{z\to t^{\pm}} \Phi_1^{\pm}(z), \quad t \in \mathbb{R}_+$$

  in the form of a coupled pair of nonhomogeneous Hilbert BVPs

- The Hilbert BVPs decouple and their solutions lead to integral equations for $\Phi_0^{\pm}(t)$ and $\Phi_1^{\pm}(t)$, $t \in \mathbb{R}_+$

- Solutions of these equations determine $\Phi_0(z)$ and $\Phi_1(z)$ on the whole cut plane $\mathbb{C} \setminus \mathbb{R}_+$ (and in turn the Laplace transform $\widehat{\varphi}(z)$)

- Removal of the poles gives algebraic constraints on Φ_0 and Φ_1

Sketch of proof

Reduction

At this stage the original eigenproblem reduces to a system of coupled integral and algebraic equations !

Sketch of proof

- The integro-algebraic system has countably many solutions, whose structure is revealed asymptotically as the algebraic part of the solution tends to $+\infty$
- The eigenvalue asymptotics is extracted from the algebraic part of the solutions
- The eigenfunctions asymptotics is obtained by Laplace transform inversion of the integral part of the solutions

Work in progress

- Spectrum of weakly singular operators, in particular with logarithmic singularities
- Spectrum of fBm type processes(fractional Brownian sheet, multidimensional case. . .) and applications
- Estimation of H in the mixed fBm processes and the standard filtering setting

Some references

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