Attractors of dissipative homeomorphisms of the infinite surface homeomorphic to a punctured sphere

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joint work with R. Ortega and A. Ruiz-Herrera. 8 ECM, minisymposium **Topological methods in dynamical systems**

Motivation:

Several results in the literature have provided sufficient conditions to guarantee a simple structure of the attractor. On the other hand, there are not so much results in the opposite direction. Our work was inspired by **Fumio Nakajima** who gave an interesting result that relates the local behavior of a dynamical system with the complexity of the attractor.

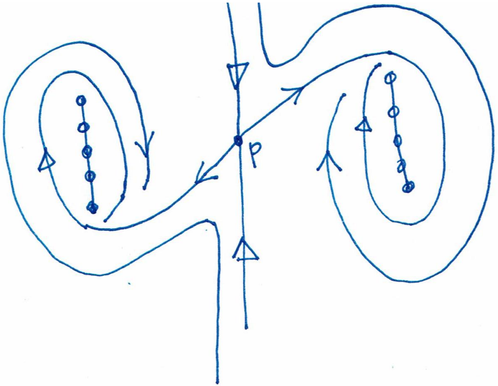
F. Nakajima, *Connected and not arcwise connected invariant sets for some 2-dimensional dynamical systems*, J. Math. Kyoto Univ. 49 (2009), no. 2, 339–346.

The result of Nakajima

Consider a smooth planar dynamical system f having two distinct fixed points, one of which is an inverse saddle. By an inverse saddle p we understand a fixed point p such that the eigenvalues of $Df(p)$, λ_1 and λ_2, satisfy

$$\lambda_1 < -1 < \lambda_2 < 0.$$

Nakajima proved that: **the existence of an inverse saddle (under some additional assumptions) implies that the attractor is not arcwise connected.**


p

The class of considered maps

We will consider self-maps of the infinite surface homeomorphic to a s-punctured sphere M_s.

Definition

We call a homeomorphism of M_s dissipative if there exists a compact set that attracts uniformly all compact sets.
For a dissipative homeomorphism $h : M_s \to M_s$ we consider the attractor $\mathcal{A} \subset M_s$, defined as the maximal compact invariant set. This set always exists and is a non-empty continuum. In the giving setting the attractor $\mathcal{A}$ may also be equivalently defined as the set of all bounded (forward and backward) orbits.

The class of considered maps

Definition

We will call a homeomorphism $h : M_s \to M_s$ a Levinson homeomorphism if it is dissipative and $\mathcal{A}$ has an empty interior.

Let us remark that Levinson homeomorphisms contain the important class of dissipative homeomorphisms contracting some Borel measure on M_s.

The case of Cylinder ($s = 2$)

What is the maximum number of Jordan curves in the attractor of Levinson homeomorphisms on the Cylinder?

Lemma

For a Levinson homeomorphisms on the Cylinder the attractor $\mathcal{A}$ contains either none Jordan curve or one Jordan curve that is not contractible.

Proof. Let $R(\Gamma_1, \Gamma_2)$ be the union of the bounded components of $C \setminus (\Gamma_1 \cup \Gamma_2)$. It turns out that all orbits in $R(\Gamma_1, \Gamma_2)$ are bounded and thus $R(\Gamma_1, \Gamma_2) \subset \mathcal{A}$, which contradicts the fact that $\mathcal{A}$ has empty interior.

Theorem

Let h be a Levinson homeomorphism of a Cylinder. Assume that h is isotopic to identity and has an inverse saddle p. **Then the attractor $\mathcal{A}$ is not arcwise connected.**

Sketch of the proof. 1. First we prove that there exist another fixed point q.

2. Next, we assume contrary to our claim that $\mathcal{A}$ is arcwise connected, and consider an arc γ joining p and q.

3. We consider $h(\gamma)$ and prove that there is a Jordan curve Γ in $\gamma \cup h(\gamma)$.

4. Due to the fact that p is an inverse saddle there must be another Jordan curve $h(\Gamma)$, thus we get two Jordan curves and we have the contradiction with Lemma.

From a more applied point of view, the above Theorem offers new dynamical insights on the global attractor in some classical models coming from non-conservative mechanics.
This perspective of our results is related to Rogerio Martins' work who proved that the attractor of the Poincaré map associated with the pendulum equation with friction **is not homeomorphic to** $\mathbb{S}^1$ provided there exists an inverse unstable fixed point. Under slightly more restrictive assumptions our theorem guarantees a stronger property: that **the attractor is not arcwise connected**.

The counterpart of mentioned above Theorem for the space M_s, where $s > 2$.

Let us denote by $\mathrm{Mod}(M_s)$ the mapping class group of M_s i.e. the group of isotopy classes of orientation-preserving homeomorphisms of M_s.

It is convenient to interpret topologically M_s as the sphere $\mathbb{S}^2$ with punctures.
Notice that two homeomorphisms h_1, h_2 belong to the same isotopy class if the behavior of their extensions $\widetilde{h}_1, \widetilde{h}_2$ coincide on the punctures. In other words, $\widetilde{h}_1$ permutes the punctures in the same manner as $\widetilde{h}_2$.

We denote by $T(h)$ the permutation of the punctures mentioned above.

Theorem

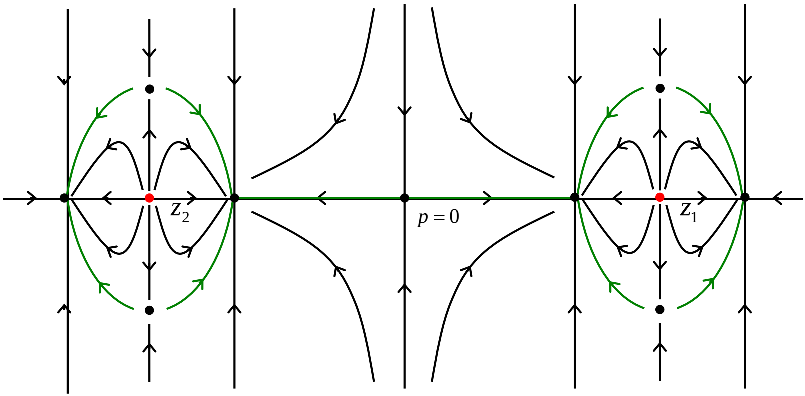
Let $h \in \mathcal{LH}(M_s)$, $s > 2$. *Assume that* h *has an inverse saddle* p *and that there is another fixed point* $q \neq p$ *of* h. *Assume also that* $T(h)$ *is a product of disjoint odd cycles. Then the attractor* $\mathcal{A}$ *is not arcwise connected.*

Remark

Theorem 5 does not hold if the assumption that $T(h)$ is a product of odd cycles is dropped.

Counterexample on M_3

Let us take $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$, $z_1 = (1, 0)$, $z_2 = (-1, 0)$, $z_3 = \infty$.
We interpret M_3 as $\mathbb{R}^2 \setminus \{z_1, z_2\}$.



Thank you for your attention!

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