Quasilinear conservation laws with discontinuous flux as singular limit of semilinear parabolic equations [1]

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[1] Joint works with Alberto Bressan and Wen Shen, Penn State University

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    - the map $x \mapsto f(x,\omega)$ is in $\boldsymbol{L}^\infty(\mathbb{R},\mathbb{R})$ for any $\omega \in \mathbb{R}$ (for parabolic approximations);
    - $f(x,\omega) = \begin{cases} f^L(\omega) & \text{for } x \leq 0, \\ f^R(\omega) & \text{for } x > 0, \end{cases}$ (singular parabolic to hyperbolic limit)

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- It models physical systems where there are conserved quantities in rough non homogeneous media:
  - elasticity theory (Keyfitz and Kranzer 1980),
  - water flooding oil recovery (Temple and Isaacson 1982; Risebro and collaborators from 1987),
  - traffic flow with discontinuities in road conditions.

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We find u^ε using a classical theorems in non-linear functional analysis.

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Definition

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- For **accretive** operators, Backward Euler is the right choice

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Backward Euler (Resolvent) operator: $J_\lambda = (I + \lambda A)^{-1}$

Given $\lambda > 0$, we say that $u = J_\lambda w$ if and only if

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- $\left[\left(I + \frac{t}{n}A\right)^{-1}\right]^n \bar{u} = \left[J_{\frac{t}{n}}\right]^n \bar{u}$

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- *For every initial datum $\bar{u} \in \overline{\mathcal{D}(A)}$ and every $t \geq 0$ the limit*

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- *The family of operators $\{S_t\, ;\ t \geq 0\}$ is a continuous semigroup of contractions on the set $\overline{\mathcal{D}(A)}$.*

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Theorem (G. G. & W. Shen, SIMA 2019, $\varepsilon > 0$ fixed)

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- $\overline{\mathcal{D}(A^\varepsilon)} = \mathbf{L}^1(\mathbb{R}, \mathbb{R})$ *density of the domain.*

Crandall–Liggett Theorem $\Longrightarrow$

Theorem (G. G. & W. Shen, SIMA 2019)

The operator A^{ε} *generates a non linear continuous semigroup* $S_t^{\varepsilon} : \boldsymbol{L}^1(\mathbb{R},\mathbb{R}) \to \boldsymbol{L}^1(\mathbb{R},\mathbb{R})$ *of contractions. The trajectory of the semigroup* $u(t,x) = [S_t^{\varepsilon}\bar{u}](x)$ *is a mild solution to the Cauchy problem*

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- The limit as $\varepsilon \to 0$ can by studied in the simpler (ODE) resolvent equation:

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- *then*

$$\lim_{\varepsilon \to 0} S_t^\varepsilon u = S_t u \qquad \forall t \geq 0,$$

and the limit is uniform for t in bounded intervals.

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$$u^\varepsilon(t,x) = -t\frac{1}{\sqrt{\varepsilon t}}\Phi\left(\frac{x}{\sqrt{\varepsilon t}}\right) \overset{*}{\rightharpoonup} -t\delta_0(x), \qquad \text{as } \varepsilon \to 0$$

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The vanishing viscosity limit for the resolvent equation

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$$D \ \doteq \ \left\{ w \in \boldsymbol{L}^1(\mathbb{R},\mathbb{R}) : \ 0 \leq w \leq 1 \right\}, \quad \text{so that}$$

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- *J_λ is the resolvent of an* accretive *map A:*

$$\mathcal{D}(A) = \bigcup_{\lambda>0} J_\lambda(D) \text{ and } Au = [f(x,u)]_x \in \mathrm{L}^1(\mathbb{R},\mathbb{R}).$$

Brézis & Pazy $\Longrightarrow$ The Limit Semigroup

Theorem (G. G. & W. Shen, SIMA 2019)

The map A previously defined generates a unique continuous semigroup of contractions $S_t : D \to D$ whose trajectories are distributional solutions to

$$u_t + [f(x,u)]_x = 0.$$

Moreover, let $S_t^\varepsilon : D \to D$ be the semigroup generated by A^ε whose trajectories are solutions to

$$u_t + [f(x,u) - \varepsilon u_x]_x = 0,$$

then the following limit holds for all $\bar{u} \in D$ uniformly on bounded t intervals.

$$S_t \bar{u} = \lim_{\varepsilon \to 0} S_t^\varepsilon \bar{u},$$

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then u^ε converges weakly to a unique limit u which is a weak solution to the limit conservation law with discontinuous flux

$$u_t + [f(v(x), u)]_x = 0$$

References

- G. G. and Wen Shen. "Vanishing Viscosity and Backward Euler Approximations for Conservation Laws with Discontinuous Flux". *SIAM J. Math. Anal.* **51** (2019), no. 4, 3112–3144.
- Alberto Bressan, G. G. and Wen Shen. "Vanishing Viscosity Solutions for Conservation Laws with Regulated Flux". *J. Differential Equations* **266** (2019), no. 1, 312–351.

Thank you for the attention!