Reverse superposition estimates, lifting over a compact covering and extensions of traces for fractional Sobolev mappings

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Reverse superposition estimates, lifting and extensions for fractional Sobolev mappings

Reverse superposition estimate for the absolute value

Lifting Sobolev maps

Extending Sobolev maps from the boundary

Absolute value and first order-Sobolev functions

A function and its absolute value have the same Sobolev semi-norm

Proposition

If $u \in W^{1,p}(\mathbb{R}^m, \mathbb{R})$, then $|u| \in W^{1,p}(\mathbb{R}^m, \mathbb{R})$ and

$$\int_{\mathbb{R}^m} |D|u||^p = \int_{\mathbb{R}^m} |Du|^p.$$

Direct and reverse fractional superposition estimate

The Gagliardo semi-norm of the absolute value controls the Gagliardo semi-norm

Proposition

If $0 < s < 1, p \geq 1$ and $u \in W^{s,p}(\mathbb{R}^m, \mathbb{R})$, then

$$[|u|]_{W^{s,p}(\mathbb{R}^m)} \leq [u]_{W^{s,p}(\mathbb{R}^m)}.$$

Proposition (JVS)

If $0 < s < 1, p \geq 1, sp > 1$ and $u \in W^{s,p}(\mathbb{R}^m, \mathbb{R})$, then

$$[u]_{W^{s,p}(\mathbb{R}^m)} \leq C[|u|]_{W^{s,p}(\mathbb{R}^m)}.$$

$$[u]_{W^{s,p}(\mathbb{R}^m)} \triangleq \iint_{\mathbb{R}^m \times \mathbb{R}^m} \frac{|u(y) - u(x)|^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x$$

Oscillation estimate

Proposition (Mironescu & JVS)

$$\iint_{\mathbb{R}^m\times\mathbb{R}^m} \frac{\operatorname{ess\,sup}_{z,w\in[x,y]} |u(z)-u(w)|^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C \iint_{\mathbb{R}^m\times\mathbb{R}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x$$

Ingredients of proof

$$\operatorname{ess\,sup}_{z,w\in[x,y]} |u(z)-u(w)| \le C[u]_{W^{s,p}([x,y])} |y-x|^{s-\frac{1}{p}}, \qquad 1 < \sigma p$$

$$\iint_{\mathbb{R}^m\times\mathbb{R}^m} \frac{[u]^p_{W^{\sigma,p}([x,y])}}{|y-x|^{m+1+(s-\sigma)p}}\,\mathrm{d}y\,\mathrm{d}x = \frac{2[u]^p_{W^{s,p}(\mathbb{R}^m)}}{(s-\sigma)p((s-\sigma)p+1)}, \qquad \sigma < s$$

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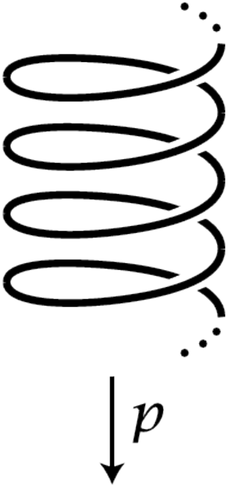
Lifting continuous maps into the circle



Problem

Can every map $u \in C(\mathcal{M}, \mathbb{S}^1)$ *be written as*

$$u = (\cos \tilde{u}, \sin \tilde{u}).$$

for some $\tilde{u} \in C(\mathcal{M}, \mathbb{R})$*?*

Theorem

If $\mathcal{M}$ *is simply-connected, then every* $u \in C(\mathcal{M}, \mathbb{S}^1)$ *can be written as*

$$u = (\cos \tilde{u}, \sin \tilde{u}).$$

for some $\tilde{u} \in C(\mathcal{M}, \mathbb{R})$.

Sobolev spaces between manifolds

Nonlinear functional spaces arising in geometry and physics

Definition

If $s \in (0,1]$ and $N \subset \mathbb{R}^\nu$ is compact,

$$W^{s,p}(\mathbb{B}^m, \mathcal{N}) = \left\{ u \in W^{s,p}(\mathbb{B}^m, \mathbb{R}^\nu) \;\middle|\; u \in \mathcal{N} \text{ almost everywhere} \right\}$$

When $s = 1$,
$u \in W^{1,p}(\mathbb{B}^m, \mathbb{R}^\nu)$
whenever u is weakly differentiable and

$$[u]^p_{1,p} \triangleq \int_{\mathbb{B}^m} |Du|^p < +\infty.$$

When $0 < s < 1$,
$u \in W^{s,p}(\mathbb{B}^m, \mathbb{R}^\nu)$ whenever

$$[u]^p_{s,p} \triangleq \int_{\mathbb{B}^m} \int_{\mathbb{B}^m} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y < +\infty.$$

Lifting Sobolev maps into the circle
The complete picture

Question

Can every $u \in W^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$ be written as

$$u = (\cos\varphi, \sin\varphi),$$

for some $\varphi \in W^{s,p}(\mathbb{B}^m, \mathbb{R})$?

Answer

F. Bethuel and Zheng X., 1988
Bourgain, Brezis & Mironescu, 2000

Yes if

- $sp \geq m$,
- $s = 1$ and $p \geq 2$,
- $sp < 1$.

Topological obstruction

If $u(x) = \frac{(x_1, x_2)}{|(x_1, x_2)|}$, then

- if $sp < 2$, then $u \in W^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$,
- if $sp \geq 1$, then u has no lifting in $W^{s,p}(\mathbb{B}^m, \mathbb{R})$.

Analytical obstruction

If $u(x) \triangleq \left(\cos\frac{1}{|x|^\alpha}, \sin\frac{1}{|x|^\alpha}\right)$, then

- if $\alpha < \frac{1}{s}(\frac{m}{p} - s)$, then $u \in W^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$,
- if $\alpha \geq \frac{m}{p} - s$, then u has **no** lifting in $W^{s,p}(\mathbb{B}^m, \mathbb{R})$.

Lifting of continuous maps over a general covering

Definition

$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a covering if for every $x \in \mathcal{N}$ there exists an open set $U \ni x$ such that $\pi^{-1}(U)$ is a union of disjoint open sets mapped homeomorphically by π on U.

- $\pi : \mathbb{R} \to \mathbb{S}^1, \pi(t) = (\cos t, \sin t)$
- $\pi : \mathbb{S}^1 \to \mathbb{S}^1, \pi(\cos t, \sin t) = (\cos kt, \sin kt)$
- $\pi : \mathbb{S}^n \subset \mathbb{R}^{n+1} \to \mathbb{RP}^n \subset \mathbb{R}^{(n+1)\times(n+1)}, \pi(x) = x \otimes x$

Theorem

If $\mathcal{M}$ is simply-connected, then every $u \in C(\mathcal{M}, \mathcal{N})$ can be written as

$$u = \pi \circ \widetilde{u}$$

with $\widetilde{u} \in C(\mathcal{M}, \widetilde{\mathcal{N}})$.

Lifting Sobolev maps over a general covering

Almost the complete picture

Question

Can every $u \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$ be written as

$$u = \pi \circ \widetilde{u},$$

with $\varphi \in W^{s,p}(\mathbb{B}^m, \widetilde{\mathcal{N}})$?

Answer (Bethuel & Chiron, 2007)

Yes if

- $s = 1$ and $p \geq 2$,
- $sp \geq m$,
- $sp < 1$.

Topological obstruction (Bethuel & Chiron, 2007)

If $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ has no lifting and $u(x) \triangleq \gamma\big(\frac{(x_1,x_2)}{|(x_1,x_2)|}\big)$, then

- if $sp < 2$, then $u \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$,
- if $sp \geq 1$, then u has no lifting in $W^{s,p}(\mathbb{B}^m, \widetilde{\mathcal{N}})$.

Analytical obstruction (Bethuel & Chiron, 2007)

If $\widetilde{\mathcal{N}}$ is complete but not compact, there exists a ray $\gamma : [0, +\infty) \to \widetilde{\mathcal{N}}$. If $u(x) \triangleq \pi(\gamma(\frac{1}{|x|^\alpha}))$, then

- if $\alpha < \frac{1}{s}(\frac{m}{p} - s)$, then $u \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$,
- if $\alpha \geq \frac{m}{p} - s$, then u has **no** lifting in $W^{s,p}(\mathbb{B}^m, \widetilde{\mathcal{N}})$.

Fractional Sobolev lifting of finite order

Completing the puzzle

Theorem (Mironescu & JVS)

If $2 \leq sp < m$ and if $\widetilde{\mathcal{N}}$ is compact, then every map $u \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$ has a lifting $\widetilde{u} \in W^{s,p}(\mathbb{B}^m, \widetilde{\mathcal{N}})$.

Theorem (Bethuel & Chiron, 2007, Mironescu & JVS)

If π is nontrivial, then each map $u \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$ has a lifting in $\widetilde{u} \in W^{s,p}(\mathbb{B}^m, \widetilde{\mathcal{N}})$
if and only if

- either $sp \geq m$,
- or $p \geq 2$ and $s = 1$,
- or $sp \geq 2$ and $\widetilde{\mathcal{N}}$ is compact,
- or $sp < 1$.

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The trace and extension problem

Which functions are traces of a Sobolev functions?

Classical trace theorem

$$u : \partial\mathbb{B}^{m+1} \to \mathbb{R} \text{ is the } \textbf{trace} \text{ of some function } U \in W^{1,p}(\mathbb{B}^{m+1}, \mathbb{R})$$

if and only if

$$u \in W^{1-\frac{1}{p},p}(\partial\mathbb{B}^{m+1}, \mathbb{R}).$$

Theorem

$$\operatorname{tr}\left(W^{1,p}(\mathbb{B}^{m+1}, \mathcal{N})\right) \subseteq W^{1-\frac{1}{p},p}(\partial\mathbb{B}^{m+1}, \mathcal{N})$$

When is the trace operator onto?

The trace problem for Sobolev maps
The current picture

Theorem

If $p \geq m$ and $\pi_m(\mathcal{N}) \simeq \{0\}$, then the trace operator is surjective.

Theorem (Hardt and Lin, 1987)

If $1 < p < m$ and $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1 \rfloor}(\mathcal{N}) \simeq 0$, then the trace operator is surjective.

Theorem (Petru Mironescu and JVS)

If $p \geq 3$, $\pi_1(\mathcal{N})$ is finite and $\pi_2(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1 \rfloor}(\mathcal{N}) \simeq \{0\}$, then the trace operator is surjective.

Qualitative obstruction (Hardt & Lin, 1987; Bethuel & Demengel, 1995)

If the trace operator is surjective and if $p \geq 2$, then $\pi_{\lfloor p-1 \rfloor}(\mathcal{N}) \simeq \{0\}$.

Quantitative obstruction (Bethuel, 2014)

If the trace operator is surjective and if $p \geq 2$, then $\pi_1(\mathcal{N}), \dots, \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ are finite.

Thank you for your attention