Weak Solutions for an Implicit, Degenerate Poro-elastic Plate System

Elena Gurvich and Justin T. Webster
University of Maryland, Baltimore County

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Relevant Collaborators:

- Ellie Gurvich, UMBC
- Lorena Bociu, North Carolina State University
- Sunny Čanić, University of California, Berkeley
- Boris Muha, University of Zagreb (Croatia)

Motivations and Related Work:

- Work with Bociu on biological applications for Biot
- Recent work by Mikelić et al. on poro-elastic plates (2-D poro-elastic systems)
- Multilayered FSI work by Čanić and Muha (mostly existence of weak solutions using semi-discretization approach)
- Recent work of Avalos, Geredeli, and Muha on semigroup formulations for multilayer systems

Main Reference Here

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Elena Gurvich
University of Maryland, Baltimore County
Baltimore, MD
gurv-3@umbc.edu

Justin T. Webster
University of Maryland, Baltimore County
Baltimore, MD
websterj@umbc.edu

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Abstract

We consider a recent plate model obtained as a scaled limit of the three dimensional Biot system of poro-elasticity. The result is a "2.5" dimensional linear system that couples traditional Euler-Bernoulli plate dynamics to a pressure equation in three dimensions, where diffusion acts only transversely. We allow the permeability function to be time-dependent, making the problem non-autonomous and disqualifying much of the standard abstract theory. Weak solutions are defined in the so called quasi-static case, and the problem is framed abstractly as an implicit, degenerate evolution problem. Utilizing the theory for weak solutions for implicit evolution equations, we obtain existence of solutions. Uniqueness is obtained under additional hypotheses on the regularity of the permeability function. We address the inertial case in an appendix, by way of semigroup theory. The work here provides a baseline theory of weak solutions for the poro-elastic plate, and exposits a variety of interesting related models and associated analytical investigations.

Key terms: poro-elasticity; plate; elliptic-parabolic; hyperbolic-parabolic; implicit evolution

MSC 2010: 74F10, 74K20, 76S05, 35D30, 34K32

Under Review

Motivating Scenario

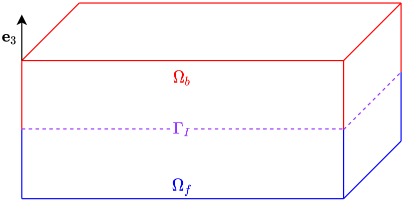
Filtration and flow of a slow-moving fluid adjacent to/through poro-elastic solid

Ω a Lipschitz domain (two stacked boxes for $x_3 \in (-1,0) \cup (0,1)$)

Top box Ω_b (Biot domain); bottom box Ω_f (Stokes domain).

The lateral variables $x_1, x_2 \in (0,1)$.

The interface $\Gamma_I = \{x_3 = 0\} \times (0,1)^2$.



Modeling Discussion:

- interface condition for Biot-Stokes coupling
- scales and regimes (inertial, compressibility)
- prevalence of engineering and homogenization-theory type work; little rigorous work

Background, Physical Questions, Mathematical Directions

- Poro-elasticity (Biot models) studied intensely last 60 years
  Often from the point of view of geoscience [Showalter et al.]
- Recently from biological tissues point of view, including *nonlinear* poro-elasticity and poro-visco-elasticity
  [Bociu et al., *ARMA* 2014]
- We are interested in:
  - Coupling conditions and regularity at Biot-Stokes interface
  - Dynamics at the interface? *Poro-elastic plate* [Mikelić et al.]
  - Perfusion through the plate (necessary)
- *Well-posedness of weak solutions for the coupled Biot-poroplate-Stokes system* [Čanić et al., 2021]

The Poro-elastic Plate

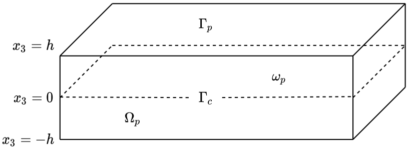
[Mikelić et al.] [Čanić et al., 2021]

ω_p is the middle surface of the poro-elastic plate at $x_3 = 0$

$(x_1, x_2) \in \omega_p$ with the transverse deflection $w(x_1, x_2; t)$

$(x_1, x_2, x_3) \in \Omega_p = \omega_p \times (-h/2, h/2)$

$x_3 \in (-h/2, h/2)$ corresponding to the transverse coordinate with $p = p(x_1, x_2, x_3; t)$



$$\begin{cases} \rho w_{tt} + D\Delta^2_{\omega_p} w + \alpha \Delta_{\omega_p} \displaystyle\int_{-h/2}^{h/2} x_3 p \, dx_3 = f(x_1, x_2, t) \quad \text{in} \quad \omega_p \\ [c_0 p - \alpha x_3 \Delta_{\omega_p} w]_t - \partial_{x_3}(k \partial_{x_3} p) = g(\mathbf{x}, t) \quad \text{in} \quad \Omega_p \\ \text{BCs, ICs} \end{cases}$$

Goal: Develop a theory of weak (strong?) solutions in the broadest *linear* sense possible.

Digression: Biot Systems

$$\Omega \subset \mathbb{R}^n; \quad \mathcal{E} \sim \text{elasticity}; \quad A(t) \sim \text{Laplacian}$$

$$\rho \boldsymbol{\eta}_{tt} + \mathcal{E}(\boldsymbol{\eta}) + \alpha \nabla p_b = \mathbf{F} \tag{1}$$

$$[c_0 p + \alpha \nabla \cdot \boldsymbol{\eta}]_t - \text{div}[\mathbf{K} \nabla p] = S \tag{2}$$

Regimes: $\rho = 0 \to$ quasi-static; $c_0 = 0 \to$ incompressible constituents

Permeability K: $A(t) = -k\Delta$; $A(t) = -\text{div}[k(x,t)\nabla \cdot]$; $A(t) = -\text{div}[k(p,\boldsymbol{\eta})\nabla \cdot]$

Cases: $\rho, c_0 > 0 \to$ thermo-elasticity; $\rho = 0 \to$ implicit; $c_0 = 0 \to$ degenerate

Quasi-static treatment: (1) Lift: $\boldsymbol{\eta} = \mathcal{E}^{-1}(\mathbf{F} - \alpha \nabla p)$; (2) Plug In to Pressure:

$$[(c_0 I + B)p]_t + A(t)p = \tilde{S}.$$

Three approaches: (1) Implicit, degenerate semigroups
(2) variational approach to weak solutions
(3) discretization methods

Major distinctions: time-dependence of permeability, nonlinearity of permeability

General Functional Setup

- V, H are separable Hilbert spaces, with $V \hookrightarrow H \hookrightarrow V'$ dense, continuous injections.
- $\mathcal{A}(t)$ is $V \to V'$ continuous and monotone for each $t \in [0, T]$.
- $\forall u, v \in V \ \ \langle \mathcal{A}(\cdot)u, v \rangle \in L^\infty(0, T)$
- $\mathcal{B}$ is self-adjoint, bounded, and monotone on H.

Definition

The family $\{\mathcal{A}(t) : V \to V' \ : \ t \in [0, T]\}$ of operators is regular if for every pair $u, v \in V$, the function $\langle \mathcal{A}(\cdot)u, v \rangle$ is absolutely continuous on $[0, T]$ and there exists a $K \in L^1(0, T)$ such that

$$\left| \frac{d}{dt} \langle \mathcal{A}(t)u, v \rangle \right| \leq K(t) \|u\|_V \ \|v\|_V, \quad u, v \in V, \text{ a.e. } t \in [0, T].$$

Abstract Problem and Weak Solutions

Suppose that $\boxed{u_0 \in H}$ and $S \in L^2(0, T; V')$ are the specified data.

Find $u \in L^2(0, T; V)$ *such that*

$$\begin{cases} \dfrac{d}{dt}[\mathcal{B}u] + \mathcal{A}(\cdot)u = S \in \ L^2(0, T; V') \\ [\mathcal{B}u](0) = \mathcal{B}u_0 \in V'. \end{cases} \tag{3}$$

Equivalently: *Find* $u \in L^2(0, T; V)$ with

$$-\int_0^T (\mathcal{B}u(t), v'(t))dt + \int_0^T \langle \mathcal{A}(t)u(t), v(t)\rangle dt = \int_0^T \langle S(t), v(t)\rangle dt + (\mathcal{B}u_0, v(0)) \tag{4}$$

holding for all

$$v \in \{w \in L^2(0, T; V) \cap H^1(0, T; H) \ : \ w(T) = 0\}.$$

Above, u is called the *weak solution* to (3).

Fundamental Results

Theorem

If there exist constants $\lambda, c > 0$ *such that*

$$2\langle \mathcal{A}(t)v, v\rangle + \lambda(\mathcal{B}v, v) \geq c||v||_V^2, \quad \forall v \in V, \quad \forall t \in [0,T],$$

then there exists a weak solution to the problem in (4)*. The particular solution satisfies*

$$||u||^2_{L^2(0,T;V)} \leq C(c,\lambda)\left[||S||^2_{L^2(0,T;V')} + (\mathcal{B}u_0, u_0)\right]. \tag{5}$$

If $\{\mathcal{A}(t) \ : \ t \in [0,T]\}$ *is a regular family of self-adjoint operators, then the solution above is unique.*

Theorem (Lions)

Let $(\mathbb{A}, \|\cdot\|_{\mathbb{A}})$ *be a Hilbert space and* $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ *be a normed linear space. If* $a : \mathbb{A} \times \mathbb{B} \to \mathbb{R}$ *is a bilinear form such that* $a(\cdot, \phi) \in \mathbb{A}'$ *for every* $\phi \in \mathbb{B}$*, then TFAE:*

- $\inf_{\|\phi\|_{\mathbb{B}}=1} \sup_{\|u\|_{\mathbb{A}} \leq 1} |a(u,\phi)| \geq c > 0,$
- *for each* $F \in \mathbb{B}'$*, there exists* $u \in \mathbb{A}$ *such that:* $a(u,\phi) = F(\phi)$ *for all* $\phi \in \mathbb{B}$*.*

If $\mathbb{B}$ *is continuously embedded in* $\mathbb{A}$ *and* a *is* $\mathbb{B}$*-elliptic, then the above Theorem holds.*

Full Interior Equations

The *inertial model*:

$$\begin{cases} \rho w_{tt} + D\Delta^2 w + \alpha\Delta \int_{-h}^{h} x_3 p \; dx_3 = f & \text{in } \omega_p, \\ [c_0 p - \alpha x_3 \Delta w]_t - \partial_3(k\partial_3 p) = g & \text{in } \Omega_p, \end{cases} \tag{6}$$

k is general—can accommodate static and time-dependent cases (linear).

We consider:

(1) compressible constituents ($c_0 > 0$) and inertial models ($\rho > 0$) when $k = const.$,

(2) quasi-static ($\rho = 0$) models when $k = k(t)$

The *quasi-static model*:

$$\begin{cases} D\Delta^2 w + \alpha\Delta \int_{-h}^{h} x_3 p \, dx_3 = f & \text{in } \omega_p, \\ [c_0 p - \alpha x_3 \Delta w]_t - \partial_3(k\partial_3 p) = g & \text{in } \Omega_p. \end{cases} \tag{7}$$

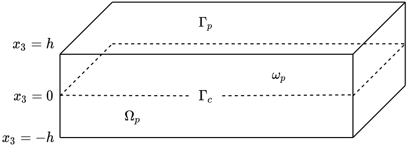
Initial and Boundary Conditions

$\underline{\rho > 0}$

$$\begin{cases} w(x_1, x_2, 0) = w_0(x_1, x_2), \quad w_t(x_1, x_2, 0) = w_1(x_1, x_2) & \text{in } \omega_p, \\ [c_0 p - \alpha x_3 \Delta w](\mathbf{x}, 0) = d_0(\mathbf{x}) & \text{in } \Omega_p. \end{cases} \tag{8}$$

$\underline{\rho = 0}$

$$[c_0 p - \alpha x_3 \Delta w](\mathbf{x}, 0) = d_0(\mathbf{x}) \ \text{ in } \Omega_p. \tag{9}$$



$$\begin{cases} w = 0; \qquad D\Delta w + \alpha \int_{-h}^{h} x_3 p \; dx_3 = 0 & \text{on } \Gamma_c, \\ k\partial_{\mathbf{n}} p = 0 & \text{on } \{x_3 = h\} \cup \{x_3 = -h\} \\ \text{Nothing on lateral sides!} \end{cases} \tag{10}$$

Quasistatic Weak Solutions

$W = H^2(\omega_p) \cap H_0^1(\omega_p) \qquad V = H^{0,0,1}(\Omega_p) = \{\phi \in L^2(\Omega_p) : \partial_{x_3}\phi \in L^2(\Omega_p)\},$

$d_0 \in V' \qquad f \in L^2(0,T;W') \qquad$ and $g \in L^2(0,T;V')$.

Definition (Quasi-static Weak Solution)

A solution to (7) with $c_0 > 0$ is represented by a pair of functions

$$w \in L^2(0,T;W) \text{ and } p \in L^2(0,T;V),$$

with $\zeta = c_0 p - x_3 \alpha \Delta w \in L^2(0,T;L^2(\Omega_p)) \cap H^1(0,T;V')$, such that:
(a) the following variational forms are satisfied for any $z \in L^2(0,T;W)$, and any $q \in \{w \in L^2(0,T;V) \cap H^1(0,T;L^2(\Omega_p)) \ : \ w(T) = 0\}$:

$$D\int_0^T (\Delta w, \Delta z)_{\omega_p} dt + \alpha \int_0^T (p, x_3 \Delta z)_{\Omega_p} dt = \int_0^T \langle f, z\rangle_{W' \times W} dt, \tag{11}$$

$$\int_0^T \left(k\partial_{x_3} p, \partial_{x_3} q\right)_{\Omega_p} dt - \int_0^T (\zeta, q_t)_{\Omega_p} \ dt = \int_0^T \langle g, q\rangle_{V' \times V} \ dt \tag{12}$$

(b) the initial condition holds in the sense of V', namely that $\lim_{t \searrow 0} \zeta(t) = d_0 \in V'$.

Main Result

Assumption

[For Existence] We assume that permeability k is an $L^\infty(\Omega_p \times (0,T))$ function such that

$$0 < k_* \leq k(\mathbf{x},t) \leq k^*, \quad \forall\, \mathbf{x} \in \Omega_p,\ \forall\, t \in [0,T].$$

Assumption

[For Uniqueness] Assume that for a.e $\mathbf{x} \in \Omega_p$ the function $k(\mathbf{x},\cdot)$ is absolutely continuous in $t \in [0,T]$, with $|\partial_t k(\mathbf{x},t)| \leq K(t)$ and $K \in L^1(0,T)$.

Theorem (Gurvich and W., *Applicable Analysis*, 2021)

Let $c_0 > 0$, and $\boxed{d_0 \in L^2(\Omega_p)}$, with $\boxed{f \in H^1(0,T;W')}$, $g \in L^2(0,T;V')$. Suppose the permeability functions k satisfies Assumption 1, then there exists a weak solution $(w,p) \in L^2(0,T;V) \times L^2(0,T;W)$. **THAT** *solution satisfies the stability estimate*

$$||p||^2_{L^2(0,T;V)} + ||w||^2_{L^2(0,T;W)} \leq C\left[||f||^2_{H^1(0,T;W')} + ||g||^2_{L^2(0,T;V')} + ||d_0||^2_{L^2(\Omega_p)}\right]. \tag{13}$$

Moreover, if k satisfies Assumption 2, then the solution is unique.

Peculiarities and Challenges of Our Model

- **Initial conditions**: Taking $w(0)$ and/or $p(0)$ and/or d_0) is a non-trivial issue, as is the space in which they are taken. When $c_0 > 0$, specifying $d_0 \in L^2(\Omega_p)$ will be equivalent to specifying $p(0) \in L^2(\Omega_p)$, and subsequently, $w(0) \in W$. In other situations, this may not be the case—see [Bociu et al., 2020 and 2021].
- **2.5-dimensional Biot Plate Equation**: Peculiar model with "inflated" domain; no full elliptic regularity in spatial component:

$$\begin{cases} D\Delta_{\omega_p}^2 w + \alpha\Delta_{\omega_p} \int_{-h/2}^{h/2} x_3 p \, dx_3 = f \quad \text{in} \quad \omega_p \\ [c_0 p - \alpha x_3 \Delta_{\omega_p} w]_t - \partial_{x_3}(k\partial_{x_3} p) = g \quad \text{in} \quad \Omega_p. \end{cases}$$

  This is problematic for plate nonlinearity—$k = k(c_0 p - \alpha x_3 \Delta w)$ or addition of f_{vK}.

- *B*-**map**: The pressure-to-Laplacian map is influenced by the BCs, which are interconnected here. "Lifting and plugging in" requires compatibility for abstract theory.

Constituent Operators

- Adjoint pair $\mathcal{K}, \tilde{\mathcal{K}}$: $\quad \mathcal{K} : L^2(\Omega_p) \to L^2(\omega_p)$ by $\quad \mathcal{K}(p) = \int_{-h}^{h} x_3 p \, dx_3$

  $\tilde{\mathcal{K}} : L^2(\omega_p) \to L^2(\Omega_p)$ by $\tilde{\mathcal{K}}(q) = x_3 q$
- For each $t \in [0, T]$, let $A(t) : H^{0,0,1}(\Omega_p) \to H^{0,0,1}(\Omega_p)'$ be defined through the bilinear form

$$A(p, q; t) = (k\partial_3 p, \partial_3 q)_{\Omega_p}, \quad \forall\, p, q \in H^{0,0,1}(\Omega_p).$$

- $\mathcal{E}$ is the hinged biharmonic operator on $L^2(\omega_p)$ with $\mathcal{E}u = \Delta^2 u$
  $\mathcal{D}(\mathcal{E}) \equiv \{w \in H^4(\omega_p) \cap H^1_0(\omega_p) \ : \ \Delta w\big|_{\Gamma_c} = 0\}$.
  $\mathcal{E}^{1/2} = (-\Delta_D)$ with $D(\mathcal{E}^{1/2}) = W = H^2(\omega_p) \cap H^1_0(\omega_p)$.
- Pressure-to-Laplacian: Denoting $\beta = \alpha^2/D$

$$Bp = \beta\tilde{\mathcal{K}}\Delta_D\mathcal{E}^{-1}\Delta_D\mathcal{K}(p) = -\alpha\tilde{\mathcal{K}}\Delta_D w = \beta\tilde{\mathcal{K}}\mathcal{K}(p).$$

  $B \in \mathscr{L}(L^2(\Omega_p))$ by the chain:

$$L^2(\Omega_p) \xrightarrow{\quad\mathcal{K}\quad} L^2(\omega_p) \xrightarrow{\quad -\alpha\Delta_D \quad} W' \xrightarrow{\quad (D\mathcal{E})^{-1} \quad} W \xrightarrow{\quad \Delta_D \quad} L^2(\omega_p) \xrightarrow{\quad -\alpha\tilde{\mathcal{K}} \quad} L^2(\Omega_p).$$

Reduction to Implicit Degenerate Equation

Abstract system:

$$\begin{cases} D\mathcal{E}(w) = -\alpha\Delta_D\mathcal{K}(p) + f & \in L^2(0,T;W'), \\ [c_0 p - \alpha\tilde{\mathcal{K}}\Delta_D w]_t - \partial_{x_3}[k\partial_{x_3} p] = g & \in L^2(0,T;V'), \\ [c_0 p - \alpha\tilde{\mathcal{K}}\Delta_D w](\mathbf{x},0) = d_0(\mathbf{x}) & \in L^2(\Omega_p). \end{cases} \quad (14)$$

Introducing $w_f(t) = D^{-1}\mathcal{E}^{-1}f(t) \in H^1(0,T;W)$ to exploit elliptic structure yields:

$$\begin{cases} D\mathcal{E}(u) = -\alpha\Delta_D\mathcal{K}(p) & \in L^2(0,T;W'), \\ [c_0 p + Bp]_t - A(t)p = g + \alpha\tilde{\mathcal{K}}\Delta_D w_{f,t} & \in L^2(0,T;V'), \\ [c_0 p + Bp](\mathbf{x},0) = d_0(\mathbf{x}) + \alpha\tilde{\mathcal{K}}\Delta_D w_f(\mathbf{x},0) & \in L^2(\Omega_p). \end{cases} \quad (15)$$

- $A(t) : H^{0,0,1}(\Omega_p) \to H^{0,0,1}(\Omega_p)'$ is self-adjoint (in that sense) and coercive ($k \geq k_* > 0$).
- $B \in \mathscr{L}(L^2(\Omega_p))$ is self-adjoint and positive.
- $c_0 I + B$ is an isomorphism on $L^2(\Omega_p)$.

Proof of Main Theorem

- Consider $V = H^{0,0,1}(\Omega_p)$ and $H = L^2(\Omega_p)$
- Let $\mathcal{B} = c_0 I + B$ on H and $\mathcal{A}(t) = A(t)$ on V.
- Coercivity: $A(p,p;t) = \langle A(t)p, p\rangle_{V' \times V}$

$$\begin{aligned} ([c_p \mathbf{I} + B]p, p)_H + 2A(p,p;t) &\geq c_p \|p\|^2_{\Omega_p} + \beta \|\mathcal{K}(p)\|^2_{\omega_p} + 2k_* \|\partial_3 p\|_{\Omega_p} \\ &\geq \min(c_p, k_*) \|p\|^2_V. \end{aligned} \tag{16}$$

- Existence is obtained, with

$$\|p\|^2_{L^2(0,T;V)} \leq C(c_p, k_*) \left[\|g\|^2_{L^2(0,T;V')} + \|\alpha \tilde{\mathcal{K}} \Delta_D w_f\|^2_{L^2(0,T;V')} + (d_0, p(0)) + \left(\alpha \tilde{\mathcal{K}} \Delta_D w_f(0), p(0)\right) \right]. \tag{17}$$

- By hypothesis:

$$\left| \frac{d}{dt} A(p,q;t) \right| \leq \int_{\Omega_p} |\partial_t k_p(\mathbf{x},t)| \, |\partial_3 p| \, |\partial_3 q| \, d\mathbf{x} \leq K(t) \|p\|_V \|q\|_V, \quad \forall \, p, q \in V, \ \forall \, t \in [0$$

Thus, $\{A(t) : V \to V' : t \in [0,T]\}$ is a regular family.
Uniqueness is obtained.

Comments

- $[c_p\mathbf{I} + B]$ is an isomorphism on $L^2(\Omega_p)$, specifying $d_0 \in L^2(\Omega_p)$ is equivalent to specifying $p(0) \in L^2(\Omega_p)$. Moreover, there exist constants such that

$$c||d_0||_{L^2(\Omega_p)} \leq ||p(0)||_{L^2(\Omega_p)} \leq C||d_0||_{L^2(\Omega_p)}. \tag{18}$$

- The original body force f is encoded through the translation, hence the time-regularity requirement:

$$g + \alpha\tilde{\mathcal{K}}\Delta_D w_{f,t} \in L^2(0, T; V')$$

- Collecting estimates we have:

$$\begin{aligned}
||w||_W &\leq C\left(\frac{\alpha}{D}\right)||\mathcal{K}p||_{L^2(\omega_p)} + ||\Delta_D^{-1}f||_{L^2(\omega_p)} \leq \ C\left[||p||^2_{L^2(\Omega_p)} + ||f||_{W'}\right], \\
||p(0)||_{L^2(\Omega_p)} &\leq C||[c_0\mathbf{I} + B]p(0)||_{L^2(\Omega_p)} \leq C||d_0||_{L^2(\Omega_p)}, \\
||\alpha\tilde{\mathcal{K}}\Delta_D w_f||_{V'} &\leq \frac{\alpha}{D}||\tilde{\mathcal{K}}\Delta_D\mathcal{E}^{-1}f||_{V'} \leq C||\tilde{\mathcal{K}}\Delta_D^{-1}f||_{L^2(\Omega_p)} \leq C||\Delta_D^{-1}f||_{L^2(\omega_p)} \leq C||f||_{W'}, \\
||f\big|_{t=0}||_{W'} &\leq C||f||_{H^1(0,T;W')}.
\end{aligned}$$

which yields the final estimate

$$||p||^2_{L^2(0,T;V)} + ||w||^2_{L^2(0,T;W)} \leq C\left[||f||^2_{H^1(0,T;W')} + ||g||^2_{L^2(0,T;V')} + ||d_0||^2_{L^2(\Omega_p)}\right]. \tag{19}$$

Future Directions

- Quasi-static strong solutions: that $\Delta_D\mathcal{E}^{-1}\Delta_D \in \mathscr{L}(L^2(\omega_p))$ was critical in the use of the abstract theory of weak solutions. The action of B extends to $U \equiv \left\{p \in L^2(\Omega_p) \ : \ \mathcal{K}p \in \mathcal{D}(\Delta_D)\right\}$

$$U \xrightarrow{\mathcal{K}} \mathcal{D}(\Delta_D) \xrightarrow{-\alpha\Delta_D} L^2(\omega_p) \xrightarrow{(D\mathcal{E})^{-1}} \mathcal{D}(\mathcal{E}) \xrightarrow{\Delta_D} \mathcal{D}(\Delta_D) \xrightarrow{-\alpha\tilde{\mathcal{K}}} U.$$

- Other configurations: Rework the abstract theory for these specific lifts
- Plate nonlinearities: permeability $k = k(c_0p + Bp)$
  von Karman/Berger for large deflections (issues with pressure modeling)
- In-plane dynamics and tangential perfusion
  [Mikelić, et al.] (decoupled if linear)
- Obtaining result for $c_0 = 0$ through singular limit [Bociu et al., *JDE* 2021]

Inertial Semigroup Result I

Consider the inertial Biot plate as before with $c, \rho > 0$, after re-scaling, with $k = const.$:

$$\begin{cases} w_t - v = 0 & \text{in } \omega_p \\ v_t + D\Delta^2 w + \alpha_p \Delta \int_{-h}^{h} x_3 p \, dx_3 = f & \text{in } \omega_p, \\ \partial_t p - \alpha_p x_3 \Delta v - \partial_3(k \partial_3 p) = g & \text{in } \Omega_p, \\ w(x_1, x_2, 0) = w_0(x_1, x_2), \quad w_t(x_1, x_2, 0) = w_1(x_1, x_2) & \text{in } \omega_p, \\ p(\mathbf{x}, 0) - \alpha x_3 \Delta w_1(x_1, x_2) = d_0(\mathbf{x}) & \text{in } \Omega_p, \\ w = 0; \quad D\Delta w + \alpha \int_{-h}^{h} x_3 p \; dx_3 = 0 & \text{on } \Gamma_c, \\ \partial_{\mathbf{n}} p = 0 & \text{on } \{x_3 = h\} \cup \{x_3 = -h\}. \end{cases} \tag{20}$$

$A = -k\partial_3^2$ defined on

$$\mathcal{D}(A) = \left\{ u \in H^{0,0,2}(\Omega_p) \; : \gamma_1[u] = 0 \text{ on } \{x_3 = \pm h\} \right\}.$$

We utilize the spaces as before:

$$\mathcal{D}(\mathcal{E}) = \{ w \in H^4(\omega_p) \cap H_0^1(\omega_p) \; : \; \gamma_0[\Delta w] = 0 \} = \mathcal{D}(\Delta_D^2), \tag{21}$$

$$\mathcal{D}(\mathcal{E}^{1/2}) = W = H^2(\omega_p) \cap H_0^1(\omega_p); \qquad V = H^{0,0,1}(\Omega_p). \tag{22}$$

Inertial Semigroup Result II

Theorem (Gurvich and W., *Applicable Analysis*, 2021)

Consider the inertial plate system (20) with $f = g = 0$*, in* $\mathbf{y} = [w, v, p]$*, posed on*

$$X = \mathcal{D}(\mathcal{E}^{1/2}) \times L^2(\omega_p) \times L^2(\Omega_p)$$

endowed with the norm $\|\mathbf{y}\|_X^2 = \left\|\mathcal{E}^{1/2} w\right\|_{0,\omega_p}^2 + \|v\|_{0,\omega_p}^2 + \|p\|_{0,\Omega_p}^2$. *Define the matrix operator*

$$\mathbf{A} : \mathcal{D}(\mathbf{A}) = \Big\{ [w, v, p] \in W \times W \times \mathcal{D}(A) \ : \ [\Delta w + \alpha \mathcal{K} p] \in \mathcal{D}(\Delta_D) \Big\}$$

with differential action

$$\mathbf{A}\mathbf{y} = \begin{pmatrix} v \\ -\mathcal{E}^{1/2}\left[\mathcal{E}^{1/2} w + \alpha \mathcal{K} p\right] \\ \alpha \tilde{\mathcal{K}} \Delta_D v - A p \end{pmatrix}, \quad \forall\, \mathbf{y} \in \mathcal{D}(\mathbf{A}).$$

Then $\mathbf{A}$ *is the generator of a strongly continuous semigroup* $\{e^{\mathbf{A}t} \ : \ t \geq 0\}$ *of contractions on* X *corresponding to the Cauchy problem.*

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}; \quad \mathbf{y}(0) = [w_0, w_1, p] \in \mathcal{D}(\mathbf{A}).$$

Thank You! And Principal References

Auriault, J.L. and Sanchez-Palencia, E., 1977. Etude du comportement macroscopique d?un milieu poreux saturé déformable. *Journal de mécanique*, 16(4), pp.575-603.

Bociu, L., Guidoboni, G., Sacco, R. and Webster, J.T., 2016. Analysis of nonlinear poro-elastic and poro-visco-elastic models. *ARMA*, 222(3), pp.1445–1519.

Bociu, L. and Webster, J.T., 2020. Nonlinear quasistatic poroelasticity. *JDE*, 296, pp.242–278.

Bociu, L., Muha, B., and Webster, J.T., Weak solutions and estimates for nonlinear Biot dynamics with incompressible constituents, 2021.

Bociu, L., Čanić, S., Muha, B. and Webster, J.T., 2021. Multilayered Poroelasticity Interacting with Stokes Flow. *SIMA*.

Marciniak-Czochra, A. and Mikelić, A., 2015. A rigorous derivation of the equations for the clamped Biot-Kirchhoff-Love poroelastic plate. *ARMA*, 215(3), pp.1035-1062.

Muha, B. and Čanić, S., 2014. Existence of a solution to a fluid multi-layered-structure interaction problem. *JDE*, 256(2), pp.658–706.

Showalter, R.E., 2000. Diffusion in poro-elastic media. *JMAA*, 251(1), pp.310–340.

Showalter, R.E., 1974. Degenerate evolution equations and applications. Indiana University Mathematics Journal, 23(8), pp.655-677.

Showalter, R.E., 2013. Monotone operators in Banach space and nonlinear partial differential equations (Vol. 49). American Mathematical Soc..

Ženíšek, A., 1984. The existence and uniqueness theorem in Biot's consolidation theory. *Aplikace matematiky*, 29(3), pp194–211.