SPECTRAL ANALYSIS OF A CONFINEMENT MODEL IN RELATIVISTIC QUANTUM MECHANICS

Albert Mas [Universitat Politècnica de Catalunya]

Joint work with
- Naiara Arrizabalaga [UPV / EHU]
- Tomás Sanz-Perela [U. Edinburgh]
- Luis Vega [UPV/EHU & BCAM]

MOTIVATION

Faber-Krahn inequality: $\Omega \subset \mathbb{R}^d$ open & bounded, $\Omega^* \subset \mathbb{R}^d$ ball with $|\Omega^*| = |\Omega|$, $\lambda_1(\Omega)$ lowest eigenvalue of the Dirichlet Laplacian $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$\longrightarrow$ $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.

[Among all membranes of a given area and with fixed boundary, the circular one produces the lowest fundamental tone.]

Interpreting
- Laplacian $\longrightarrow$ free Schrödinger operator
- Dirichlet boundary condition $\longrightarrow$ hard-wall boundary

[The ground-state energy of a nonrelativistic quantum particle constrained to nanostructures of a given material is minimized by a ball.]

In quantum mechanics:

- $-\Delta$ nonrelativistic $\longrightarrow$ Dirac operator { relativistic (& local!) }
- Dirichlet boundary condition $\longrightarrow$ MIT bag boundary condition (also called "infinite mass" boundary condition)

DIRAC OPERATORS

$H := -i\alpha\cdot\nabla + m\beta$
$\quad = -i(\alpha_1\partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3) + m\beta$

$\Omega \subset \mathbb{R}^3$

- $m \geq 0$ mass.
- $\alpha_1, \alpha_2, \alpha_3, \beta$ 4×4 matrices $\begin{cases} \text{anticommute} \\ \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = \mathrm{Id} \end{cases}$

$\Longrightarrow H^2 = (-\Delta + m^2)\mathrm{Id}$ (in the spirit of $-\Delta = \frac{1}{4}\partial_z\partial_{\bar z}$ in $\mathbb{R}^2 \equiv \mathbb{C}$)

$\Longrightarrow H$ is a local version of $\sqrt{-\Delta + m^2}$ (not semibounded)

MIT bag boundary condition : $u = -i\beta(\alpha\cdot\nu)u$ on $\partial\Omega$ (normal vector field on $\partial\Omega$)

We study the family $\{H_\tau\}_{\tau\in\mathbb{R}}$: $(\tau = 0)$

$\mathrm{Dom}(H_\tau) := \{\varphi \in H^1(\Omega;\mathbb{C}^4) : \varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi \text{ on } \partial\Omega\}$

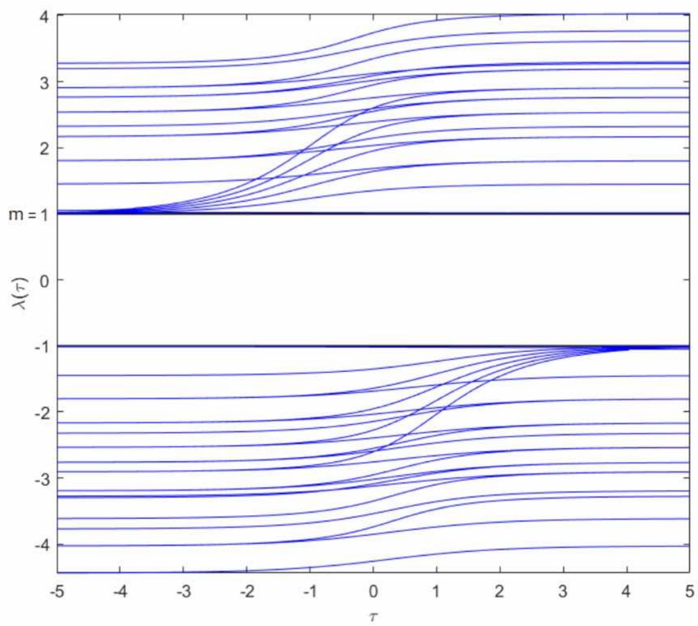
$H_\tau\varphi := (-i\alpha\cdot\nabla + m\beta)\varphi \quad \forall\, \varphi \in \mathrm{Dom}(H_\tau)$

Properties of the spectrum of H_τ:

- $\sigma(H_\tau) \subset \mathbb{R}\setminus[-m,m]$, purely discrete.
- $\lambda \in \sigma(H_\tau) \Longleftrightarrow -\lambda \in \sigma(H_{-\tau})$.

We want to DESCRIBE the map $\tau \in \mathbb{R} \longmapsto \underbrace{\lambda(\tau)}_{\text{(eigenvalues of } H_\tau)} \in \sigma(H_\tau) \cap (m, +\infty)$
[eigenvalue curves]

THE BALL



The spectral study on a ball is "explicit" (Bessel functions):

Eigenvalue curves $\tau \mapsto \lambda(\tau)$ are:

* smooth
* increasing
* $\lambda(+\infty) \in \sqrt{\sigma(-\Delta_D) + m^2}$
* $\lambda(-\infty) \in \sqrt{\sigma(-\Delta_D) + m^2} \cup \{m\}$

RESULTS

$\Omega \subset \mathbb{R}^3$ bounded domain with C^2 boundary.

* $\tau \mapsto \lambda(\tau) \in \sigma(H_\tau)$ are real analytic & increasing:

- Perturbation theory (Kato) + Krein-type resolvent formula

$$(H_\tau - \xi)^{-1} = (H_0 - \xi)^{-1} + T(\tau, \xi)$$

(holomorphic in τ)

$\Longrightarrow$ parametrization & smoothness.

- Differentiate $H\varphi = \lambda(\tau)\varphi$ + integrate by parts
  + boundary condition for $\varphi \in \mathrm{Dom}(H_\tau) \Longrightarrow \frac{d\lambda}{d\tau} > 0$.

* $\lim_{\tau \to \pm\infty} \lambda(\tau)$ & Dirichlet Laplacian $-\Delta_D$:

- Reproducing formula for $(H-\lambda)\varphi = 0$ (like Cauchy formula for $\partial_{\bar{z}} f = 0$) + regularity estimates for (fractional and singular) integral operators on $\partial\Omega$

$\longrightarrow$ uniform estimates for $\varphi \longrightarrow$ compactness as $\tau \to \pm\infty$.

- $\left.\begin{array}{l} H^2 = -\Delta + m^2 \\ H\varphi = \lambda\varphi \end{array}\right\} \Longrightarrow (-\Delta + m^2)\varphi = H^2\varphi = \lambda^2\varphi \longrightarrow -\Delta\varphi = (\lambda^2 - m^2)\varphi$.

- $\frac{d\lambda}{d\tau} > 0 \Longrightarrow \exists\, \lambda(\pm\infty) := \lim_{\tau\to\pm\infty} \lambda(\tau) \in [\underset{(\tau=-\infty)}{m}, \underset{(\tau=+\infty)}{+\infty}]$

$$\left[\varphi = i(\sinh\tau - \cosh\tau\,\beta)(\alpha\cdot\nu)\varphi \text{ on } \partial\Omega, \quad \beta = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}\right]$$

$\lim_{\tau\to\pm\infty} (\sinh\tau \mp \cosh\tau) = 0 \Longrightarrow$ Some components of φ vanish on $\partial\Omega$

$\longrightarrow$ vanishing Dirichlet condition $\longrightarrow \begin{cases} \lambda(+\infty)^2 - m^2 \in \sigma(-\Delta_D) \cup \{+\infty\} \\ \lambda(-\infty)^2 - m^2 \in \sigma(-\Delta_D) \cup \{0\} \end{cases}$

SHAPE OPTIMIZATION

(indeed open for H_τ $\forall \tau \in \mathbb{R}$)

Open problem [Faber-Krahn inequality for H_0: MIT bag model]:

"Among all $\Omega \subset \mathbb{R}^3$ with same volume, the ball has the smallest positive eigenvalue of H_0".

Set $\lambda_\Omega(\tau) :=$ "1st positive eigenvalue of H_τ on Ω".

- On a ball $B \subset \mathbb{R}^3$ we have $\lambda_B(\tau) \xrightarrow{\tau \to +\infty} \sqrt{\underbrace{\lambda_1(-\Delta_D; B)}_{\text{(1st eigenvalue of } -\Delta_D)} + m^2}$

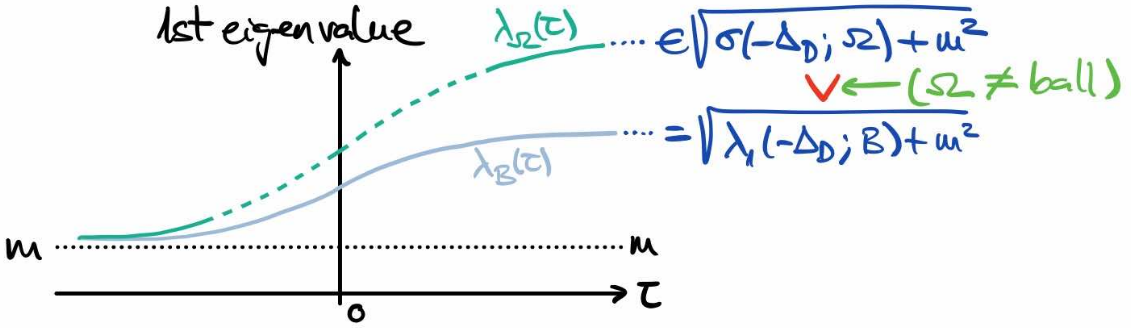
  (Bessel functions)

- For any $\Omega \subset \mathbb{R}^3$ we know that $\lambda(+\infty)^2 - m^2 \in \sigma(-\Delta_D) \cup \{+\infty\}$.

$$\Longrightarrow \lambda_\Omega(\tau) \xrightarrow{\tau \to +\infty} \lambda_\Omega(+\infty) \geq \sqrt{\lambda_1(-\Delta_D; \Omega) + m^2}$$

$$\geq \sqrt{\lambda_1(-\Delta_D; B) + m^2} \xleftarrow{\tau \to +\infty} \lambda_B(\tau)$$

[Faber-Krahn inequality for $-\Delta_D$ on Ω and B with $|\Omega| = |B|$]



Therefore, if $\Omega \subset \mathbb{R}^3$ is not a ball, and $|B| = |\Omega|$

$\Longrightarrow \lambda_B(\tau) < \lambda_\Omega(\tau)$ for all $\tau \in \mathbb{R}$ big enough.

The case $\tau \to -\infty$ is more difficult: $\lambda_\Omega(-\infty) = m$ $\forall \Omega \subset \mathbb{R}^3$.

We expand $\lambda_\Omega(\tau) = m + \underbrace{L_\Omega}_{\text{("slope at } -\infty\text{")}} e^\tau + o(e^\tau)$ as $\tau \to -\infty$

Description of L_Ω using a Rayleigh quotient:

$$\frac{1}{L_\Omega} \leq \sup_{u \in \text{Hardy space on } \partial\Omega} \frac{\int_{\partial\Omega} [\text{Single layer of } -\Delta] u \cdot u}{\int_{\partial\Omega} |u|^2}.$$

[like "traces on $\partial\Omega$ of functions holomorphic on Ω"]