EXISTENCE RESULTS OF FOURTH ORDER EQUATIONS WITH PERTURBED TWO-POINT BOUNDARY CONDITIONS

Alberto Cabada

(Joint work with Rochdi Jebari)

Departamento de Estatística, Análise Matemática e Optimización
Universidade de Santiago de Compostela, Galicia,
SPAIN

8th European Congress of Mathematics

Special Session on Topological Methods in Differential Equations

June 24th, 2021

ABSTRACT

In this talk we establish the existence and multiplicity of positive solutions for a fourth-order boundary value problem.

Integral perturbations of some kind of two-point boundary conditions are considered.

After the construction of a Green's function and the study of its constant sign, it is defined a positive cone, where to apply the Krasnoselskii compression/expansion and Leggett-Williams fixed point theorems in cones.

A generalization for a higher order case is also considered. Some particular examples are given.

ABSTRACT

The results are compiled on the paper

R. JEBARI AND A. C. Multiplicity results for fourth order problems related to the theory of deformations beams *Discrete Contin. Dyn. Syst.-B*, **25**, 2, (2020), p. 489–505.

PARTS OF THE TALK

- PRELIMINARIES

PARTS OF THE TALK

- PRELIMINARIES
- GREEN'S FUNCTIONS

PARTS OF THE TALK

- PRELIMINARIES
- GREEN'S FUNCTIONS
- NONLINEAR PROBLEMS

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory
- shooting method

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory
- shooting method
- fixed point theorems on cones

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory
- shooting method
- fixed point theorems on cones
- critical point theory

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory
- shooting method
- fixed point theorems on cones
- critical point theory
- lower and upper solutions

The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam.

They have been studied by many authors via several methods, such as

- Leray-Schauder continuation method
- topological degree theory
- shooting method
- fixed point theorems on cones
- critical point theory
- lower and upper solutions
- spectral theory

R. Y. MA, J. H. ZHANG AND F. M. SHENGMAO, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.*, **215** (1997), 415–422.

R. Y. Ma, J. H. Zhang and F. M. Shengmao, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.*, **215** (1997), 415–422.

J. M. Davis and J. Henderson, Uniqueness implies existence for fourth-order Lidstone boundary value problems, *Panamer. Math. J.*, **8** (1998), 23–35.

- R. Y. MA, J. H. ZHANG AND F. M. SHENGMAO, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.*, **215** (1997), 415–422.
- J. M. DAVIS AND J. HENDERSON, Uniqueness implies existence for fourth-order Lidstone boundary value problems, *Panamer. Math. J.*, **8** (1998), 23–35.
- Z. B. BAI AND H. Y. WANG, On the positive solutions of some nonlinear fourth-order beam equations *J. Math. Anal. Appl.*, **270** (2002), 357–368.

J. Á. CID, L. SANCHEZ AND A. C. Positivity and lower and upper solutions for fourth order boundary value problems *Nonlinear Anal.*, **67** (2007), 1599–1612.

J. Á. CID, L. SANCHEZ AND A. C. Positivity and lower and upper solutions for fourth order boundary value problems *Nonlinear Anal.*, **67** (2007), 1599–1612.

G. BONANNO AND B. DI BELLA, A boundary value problem for fourth-order elastic beam equations *Nonlinear Anal.*, *J. Math. Anal. Appl.*, **343** (2008), 1166–1176.

J. Á. CID, L. SANCHEZ AND A. C. Positivity and lower and upper solutions for fourth order boundary value problems *Nonlinear Anal.*, **67** (2007), 1599–1612.

G. BONANNO AND B. DI BELLA, A boundary value problem for fourth-order elastic beam equations *Nonlinear Anal., J. Math. Anal. Appl.*, **343** (2008), 1166–1176.

J. R. GRAEF, J. HENDERSON AND B. YANG, Positive solutions to a fourth-order three point boundary value problem, *Discrete Contin. Dyn. Syst.*, (2009), 269–275.

P. DRÁBEK AND G. HOLULOVÁ Positive and negative solutions of one-dimensional beam equation *Appl. Math. Lett.*, **51** (2016), 1-7.

P. DRÁBEK AND G. HOLULOVÁ Positive and negative solutions of one-dimensional beam equation *Appl. Math. Lett.*, **51** (2016), 1-7.

P. DRÁBEK AND G. HOLULOVÁ On the maximum and antimaximum principles for the beam equation *Appl. Math. Lett.*, **56** (2016), 29-33.

C. FERNÁNDEZ-GÓMEZ AND A. C. Constant sign solutions of two-point fourth order problems *Appl. Math. Comput.*, **263** (2015), 122–133.

$$\begin{cases} u^{(4)}(t) + M\,u(t) = \sigma(t), \quad t \in I := [0,1] \\ u(0) = u'(0) = u''(0) = u(1) = 0, \end{cases} \tag{1}$$

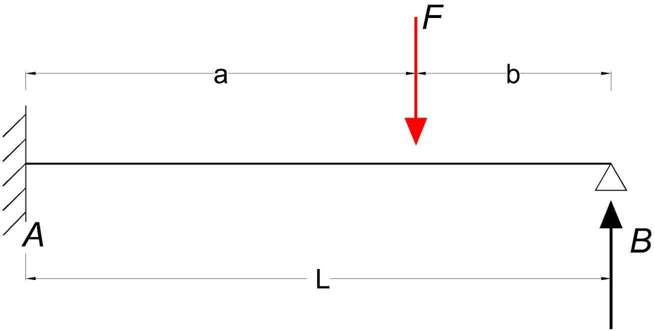


FIGURE: Beam clamped at one point and supported at the other

Engineering ToolBox, (2003). Beams - Fixed at One End and Supported at the Other - Continuous and Point Loads .

[online] Available at: https://www.engineeringtoolbox.com/beams-fixed-end-d560.html [June, 22, 2021].

After introducing the following spectral condition

$$M > 0 \quad \text{and} \quad \tan\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right) = \tanh\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right). \tag{2}$$

The following result is obtained

After introducing the following spectral condition

$$M > 0 \quad \text{and} \quad \tan\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right) = \tanh\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right). \qquad (2)$$

The following result is obtained

LEMMA

Let $\sigma \in C(I)$ and $M \in \mathbb{R}$ be such that (2) does not hold. Then problem (1) has a unique solution given by

$$u(t) = \int_0^1 g_M(t,s)\sigma(s)\,ds.$$

where $g_M(t,s)$ is the so-called Green's Function.

Let $M \in \mathbb{R}$ be such that (2) does not hold.

LEMMA

The Green's function g_M satisfies the following properties:

Let $M \in \mathbb{R}$ be such that (2) does not hold.

LEMMA

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1).$

Let $M \in \mathbb{R}$ be such that (2) does not hold.

Lemma

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1).$
2. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = \frac{\partial^2 g_M}{\partial s^2}(t,1) = g_M(t,0) = 0, t \in (0,1).$

Let $M \in \mathbb{R}$ be such that (2) does not hold.

LEMMA

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1)$.
2. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = \frac{\partial^2 g_M}{\partial s^2}(t,1) = g_M(t,0) = 0, t \in (0,1)$.

Let $M \in \mathbb{R}$ be such that (2) does not hold.

LEMMA

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1)$.
2. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = \frac{\partial^2 g_M}{\partial s^2}(t,1) = g_M(t,0) = 0, t \in (0,1)$.

LEMMA

If the Green's function $g_M(t,s) < 0$ for all $t, s \in (0,1)$, then it satisfies the following properties:

Let $M \in \mathbb{R}$ be such that (2) does not hold.

Lemma

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1).$
2. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = \frac{\partial^2 g_M}{\partial s^2}(t,1) = g_M(t,0) = 0, t \in (0,1).$

Lemma

If the Green's function $g_M(t,s) < 0$ for all $t, s \in (0,1)$, then it satisfies the following properties:

1. $\frac{\partial^3 g_M}{\partial t^3}(0,s) < 0 < \frac{\partial g_M}{\partial t}(1,s), s \in (0,1).$

Let $M \in \mathbb{R}$ be such that (2) does not hold.

LEMMA

The Green's function g_M satisfies the following properties:

1. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = \frac{\partial^2 g_M}{\partial t^2}(0,s) = g_M(1,s) = 0, s \in (0,1).$
2. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = \frac{\partial^2 g_M}{\partial s^2}(t,1) = g_M(t,0) = 0, t \in (0,1).$

LEMMA

If the Green's function $g_M(t,s) < 0$ for all $t, s \in (0,1)$, then it satisfies the following properties:

1. $\frac{\partial^3 g_M}{\partial t^3}(0,s) < 0 < \frac{\partial g_M}{\partial t}(1,s), s \in (0,1).$
2. $\frac{\partial^3 g_M}{\partial s^3}(t,1) > 0 > \frac{\partial g_M}{\partial s}(t,0), t \in (0,1).$

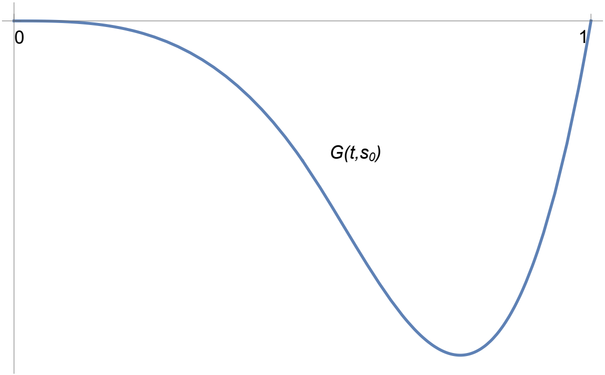


FIGURE: Graph of $G(t, s_0)$, $t \in I$

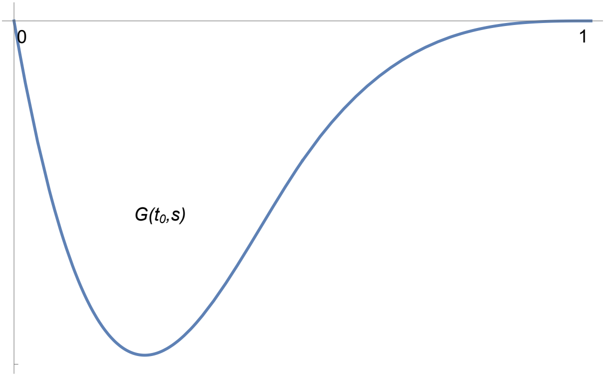


Figure: Graph of $G(t_0, s)$, $s \in I$

Moreover it is characterized the constant sign of the Green's function g_M related to the fourth order linear problem (1).

Moreover it is characterized the constant sign of the Green's function g_M related to the fourth order linear problem (1).

There, by using the Disconjugacy Theory, are obtained the exact values on the real parameter $M \in [-m_0^4, m_1^4)$, for which the related Green's function $g_M < 0$ in $(0,1) \times (0,1)$.

To be concise, $m_0 \cong 4.73004$ is the first positive root of equation

$$\cos m \cosh m = 1,$$

and $m_1 \cong 5.553$ is the first positive root of equation

$$\tan \frac{m}{\sqrt{2}} = \tanh \frac{m}{\sqrt{2}}.$$

Such result has been extended in

L. SAAVEDRA AND A. C. The eigenvalue characterization for the constant sign Green's functions of $(k, n-k)$ problems *Bound. Value Probl.*, **2016** (2016), 35 pp.

for any n-th order differential operator, coupled to the so-called $(k, n-k)$ boundary conditions, which are defined, for $1 \leq k \leq n-1$, as follows:

$$u(0) = u'(0) = \cdots = u^{(k-1)}(0) = u(1) = u'(1) = \cdots = u^{(n-k-1)}(1) = 0.$$

There is proved that $-m_1^4$ is the biggest negative eigenvalue of operator $u^{(4)}$ in the space

$$\{u \in C^4(I), \quad u(0) = u'(0) = u''(0) = u(1) = 0\}$$

There is proved that $-m_1^4$ is the biggest negative eigenvalue of operator $u^{(4)}$ in the space

$$\{u \in C^4(I), \quad u(0) = u'(0) = u''(0) = u(1) = 0\}$$

and m_0^4 is the least positive eigenvalue of operator $u^{(4)}$ on the space

$$\{u \in C^4(I), \quad u(0) = u'(0) = u(1) = u'(1) = 0\}.$$

To allow, on the boundary conditions, suitable dependence at some fixed points (or sets) of the interval, that are not the extreme ones, permits the study of a wider set of problems that model suitable real phenomena.

Therefore, the so-called non-local conditions allow us to deal with more complicated problems that model more difficult real phenomena.

In the non resonance case, such kind of problems can be studied as an equivalent integral equation of the type

$$u(t) = r(t)\,B(u) + \int_0^1 k(t,s) f(s,u(s))\,ds, \quad t \in I, \tag{3}$$

where r is a continuous function, $B : C(I) \to \mathbb{R}$ is a continuous functional, k is the Green's function related to the non local problem, and $f(t,x)$ is the nonlinear part of the considered equation.

This kind of equations cover different non-local situations as, for instance, to model the steady-state of a heated bar of length 1 subject to a thermostat, where a controller in one end adds or removes heat accordingly to the temperature measured by a sensor at a point of the bar.

Heat-flow problem has been studied in several works on the literature.

G. Infante and J. R. L. Webb, *Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations*, Proc. Edinb. Math. Soc., **49** (2006), 637–656.

Heat-flow problem has been studied in several works on the literature.

- G. Infante and J. R. L. Webb, *Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations*, Proc. Edinb. Math. Soc., **49** (2006), 637–656.
- J. R. L. Webb, *Existence of positive solutions for a thermostat model*, Nonlinear Anal. Real World Appl., **13** (2012), 923–938.

Heat-flow problem has been studied in several works on the literature.

G. Infante and J. R. L. Webb, *Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations*, Proc. Edinb. Math. Soc., **49** (2006), 637–656.

J. R. L. Webb, *Existence of positive solutions for a thermostat model*, Nonlinear Anal. Real World Appl., **13** (2012), 923–938.

G. Infante, P. Pietramala and F. A. F. Tojo, *Nontrivial solutions of local and nonlocal Neumann boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A, 146 (2016), 2, 337–369.

Motivated by the above works, in this talk we study the existence and multiplicity of positive solutions for the fourth order equation:

$$u^{(4)}(t) + M\,u(t) + f(t, u(t)) = 0, \quad t \in I, \tag{4}$$

subject to the perturbed functional boundary conditions:

$$u(0) = u'(0) = u''(0) = 0,\; u(1) = \lambda \int_0^1 u(s)\,ds. \tag{5}$$

Here f is such that

(H_0) $f : I \times [0, \infty) \to [0, \infty)$ is a continuous function, $M \in [-m_0^4, m_1^4)$ and λ is a positive parameter bounded from above by a constant that will be introduced later.

A function $u \in C(I)$ is called nonnegative solution of Problem (4)-(5) if u is a solution of (4)-(5) and $u(t) \geq 0$, for all $t \in I$. A function $u \in C(I)$ is called positive solution of Problem (4)-(5) if u is a nonnegative solution of (4)-(5) and $u(t) > 0$, for all $t \in (0, 1)$.

A standard approach to study positive solutions of a boundary value problem such as (4) – (5) consists on finding the corresponding Green's function G_M and seek solutions as fixed points of the following integral operator:

$$T_{M,\lambda}(u)(t) := \int_0^1 G_{M,\lambda}(t,s) f(s,u(s))\, ds \tag{6}$$

in the cone $P = \{u \in C(I), u \geq 0 \text{ on } I\}$ of non-negative functions in the space $C(I)$ endowed with the usual supremum norm.

Notice that we are rewritten equation (3), erasing the nonhomogeneous term and including it on the expression of the new Green's function $G_{M,\lambda}$.

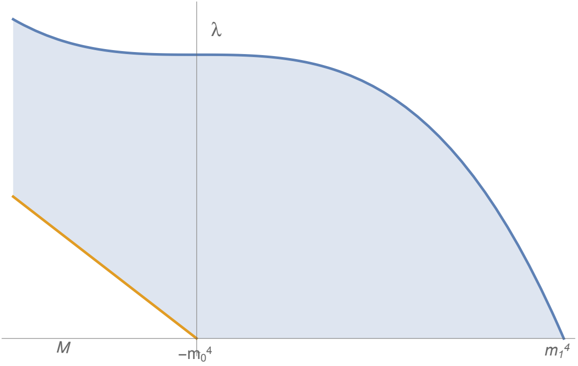
We will obtain the expression of the Green's function $G_{M,\lambda}$ related to the linear equation $u^{(4)}(t) + Mu(t) = 0$ coupled to the functional boundary conditions (5) as a combination of the expression of g_M, the Green's function related to Problem (1).

We will obtain the expression of the Green's function $G_{M,\lambda}$ related to the linear equation $u^{(4)}(t) + Mu(t) = 0$ coupled to the functional boundary conditions (5) as a combination of the expression of g_M, the Green's function related to Problem (1).

In this case, we will give the exact values on the positive parameter λ for which $G_{M,\lambda} > 0$ on $(0,1) \times (0,1)$, whenever $M \in [-m_0^4, m_1^4)$.

Moreover, we prove that there is no $\lambda > 0$ for which $G_{M,\lambda} > 0$ when $M > m_1^4$.

In the case of $M < -m_0^4$ we have that it must exists a set of values of λ where such property is fulfilled, but this case has not been considered in this paper, and remains as an open problem.



Now, we point out that problem

$$\begin{cases} u^{(4)}(t) + M u(t) = 0, & t \in I, \\ u(0) = u'(0) = u''(0) = 0, & \\ u(1) = 1, & \end{cases}$$

has no solution if and only the spectral condition (2) holds.

Now, we point out that problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = 0, & t \in I, \\ u(0) = u'(0) = u''(0) = 0, & \\ u(1) = 1, & \end{cases}$$

has no solution if and only the spectral condition (2) holds.

In any other case, it has a unique solution, denoted by w_M.

It is not difficult to verify that $w_M(t) > 0$ for all $t \in I$ if and only if $M < m_1^4$.

On the other hand, if we consider the following boundary value problem:

$$v^{(4)}(t) + M\,v(t) = \sigma(t),\ \ t \in I,\ \ v(0) = u(1) = v'(1) = v''(1) = 0, \tag{7}$$

One can see that Problem (7) is just the adjoint of Problem (1). So, the eigenvalues of both problems coincide and Green's function g_M^* related to this problem satisfies that

$$g_M^*(t,s) = g_M(s,t) \text{ for all } t, s \in I.$$

As a direct consequence, we have that

$$z_M(s) = \int_0^1 g_M^*(s,r)\,dr = \int_0^1 g_M(r,s)\,dr$$

is the unique solution of the following boundary value problem:

$$z^{(4)}(t) + M\,z(t) = 1 \quad t \in I, \quad z(0) = z(1) = z'(1) = z''(1) = 0.$$

Moreover, we have that if $M \in [-m_0^4, m_1^4)$ then $z_M(s) < 0$ for all $s \in (0,1)$, and $z_M'(0) < 0 < z_M'''(1)$.

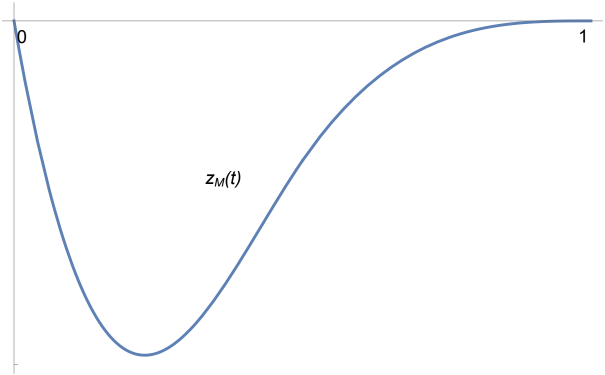


Figure: Graph of $z_M(t)$, $t \in I$.

LEMMA

Let $\sigma \in L^1(I)$, $\lambda > 0$ and $M \in \mathbb{R}$ be such that (2) does not hold. Then problem

$$\begin{cases} u^{(4)}(t) + Mu(t) + \sigma(t) = 0, \quad t \in I, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \displaystyle\int_0^1 u(s)\, ds, \end{cases}$$

has a unique solution if and only if

$$\lambda\, C_M \equiv \lambda \int_0^1 w_M(\tau)\, d\tau \neq 1.$$

Lemma

In such a case, it is given by the following expression

$$u_{M,\lambda}(t) = \int_0^1 G_{M,\lambda}(t,s)\sigma(s)\,ds$$

where

$$G_{M,\lambda}(t,s) = -g_M(t,s) - \frac{\lambda w_M(t)}{1-\lambda C_M}\int_0^1 g_M(\tau,s)\,d\tau. \qquad (8)$$

THEOREM

Let $G_{M,\lambda}(t,s)$ *be Green's function related to problem* (4)–(5).

Then if $M \in [-m_0^4, m_1^4)$ *and* $\lambda \in (0, 1/C_M)$ *we have that* $G_{M,\lambda}(t,s) > 0$ *for all* $(t,s) \in (0,1) \times (0,1)$.

THEOREM

Let $G_{M,\lambda}(t,s)$ *be Green's function related to problem* (4)–(5).

Then if $M \in [-m_0^4, m_1^4)$ *and* $\lambda \in (0, 1/C_M)$ *we have that* $G_{M,\lambda}(t,s) > 0$ *for all* $(t,s) \in (0,1) \times (0,1)$.

Theorem

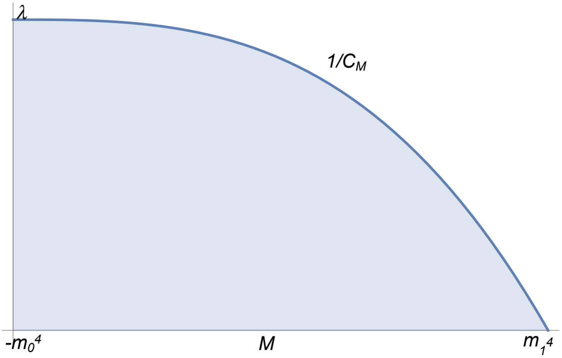
Let $G_{M,\lambda}(t,s)$ be Green's function related to problem (4)–(5).

Then if $M \in [-m_0^4, m_1^4)$ and $\lambda \in (0, 1/C_M)$ we have that $G_{M,\lambda}(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

Moreover there exist $R > 0$ and $h \in C(I)$, such that $h(0) = 0$ and $h > 0$ on $(0,1]$, for which the following inequalities are fulfilled:

$$h(t)\,\frac{\lambda}{\lambda\, C_M - 1} z_M(s) \leq G_{M,\lambda}(t,s) \leq R\,\frac{\lambda}{\lambda\, C_M - 1} z_M(s),\ t,s \in I.$$

Construction of the Green's Function



Proof.

Since $M \in [-m_0^4, m_1^4)$ we have that $g_M < 0$ and, as a direct consequence of $\lambda \in (0, 1/C_M)$ and the fact that $w_M > 0$ on I for all $M < m_1^4$, we conclude that $G_{M,\lambda}(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

Proof.

Since $M \in [-m_0^4, m_1^4)$ we have that $g_M < 0$ and, as a direct consequence of $\lambda \in (0, 1/C_M)$ and the fact that $w_M > 0$ on I for all $M < m_1^4$, we conclude that $G_{M,\lambda}(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

Proof.

Since $M \in [-m_0^4, m_1^4)$ we have that $g_M < 0$ and, as a direct consequence of $\lambda \in (0, 1/C_M)$ and the fact that $w_M > 0$ on I for all $M < m_1^4$, we conclude that $G_{M,\lambda}(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

Now, we denote by

$$\varphi(t,s) = \frac{G_{M,\lambda}(t,s)}{G_{M,\lambda}(1,s)} = \frac{1-\lambda C_M}{\lambda} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(r,s)\,dr} + w_M(t).$$

It is clear that function $\varphi(t,s)$ is continuous on $[0,1] \times (0,1)$. □

Proof.

Using the properties of g_M and those of z_M, we deduce, for all $t \in (0,1)$:

$$\lim_{s\to 0^+} \frac{g_M(t,s)}{\int_0^1 g_M(r,s)\,dr} = \lim_{s\to 0^+} \frac{g_M(t,s)}{z_M(s)} = \lim_{s\to 0^+} \frac{\frac{\partial g_M}{\partial s}(t,s)}{z_M'(s)} = \frac{\frac{\partial g_M}{\partial s}(t,0)}{z_M'(0)} > 0.$$

Thus,

$$\lim_{s\to 0^+} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda} \left(\frac{\frac{\partial g_M}{\partial s}(t,0)}{z_M'(0)} \right) + w_M(t) := l_1(t) > 0,\, t \in (0,1].$$

□

Proof.

Analogously, if $t \in (0,1)$, we have

$$\lim_{s \to 1^-} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(r,s)\,dr} = \lim_{s \to 1^-} \frac{g_M(t,s)}{z_M(s)} = \lim_{s \to 1^-} \frac{\frac{\partial^3 g_M}{\partial s^3}(t,s)}{z_M'''(s)} = \frac{\frac{\partial^3 g_M}{\partial s^3}(t,1)}{z_M'''(1)} > 0$$

and

$$\lim_{s \to 1^-} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda} \left(\frac{\frac{\partial^3 g_M}{\partial s^3}(t,1)}{z'''(1)} \right) + w_M(t) := l_2(t) > 0,\ t \in (0,1].$$

□

PROOF.

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so φ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\widetilde{\varphi} \in C(I \times I)$.

Proof.

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so φ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\widetilde{\varphi} \in C(I \times I)$.

Proof.

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so φ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\widetilde{\varphi} \in C(I \times I)$.

Therefore $h(t) = \min_{s \in [0,1]} \widetilde{\varphi}(t, s)$ is a continuous function such that

$$h(0) = 0 \quad \text{and} \quad 0 < h(t) \leq \widetilde{\varphi}(t, s) \leq R \text{ for all } (t, s) \in (0, 1] \times [0, 1],$$

where $R = \max_{(t,s) \in I \times I} \widetilde{\varphi}(t, s)$.

Proof.

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so φ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\widetilde{\varphi} \in C(I \times I)$.

Therefore $h(t) = \min_{s\in[0,1]} \widetilde{\varphi}(t, s)$ is a continuous function such that

$$h(0) = 0 \quad \text{and} \quad 0 < h(t) \le \widetilde{\varphi}(t, s) \le R \text{ for all } (t, s) \in (0, 1]\times[0, 1],$$

where $R = \max_{(t,s)\in I\times I} \widetilde{\varphi}(t, s)$.

The result follows from the expression of $G_{M,\lambda}(1, s)$. □

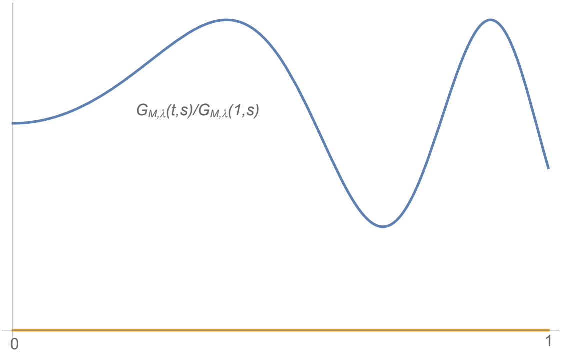


Figure: Graph of $\widetilde{\varphi}(t, s)$, $t \in I$.

COROLLARY

Let $G_{M,\lambda}(t,s)$ be Green's function related to problem (4)–(5) given by expression (8). Then if $M \in [-m_0^4, m_1^4)$ and $\lambda \in (0, 1/C_M)$ we have that for all $\delta \in (0,1)$ there exists $\gamma(\delta) \in (0,1)$ for which the following inequality is fulfilled:

$$\gamma \frac{\lambda}{\lambda\, C_M - 1} z_M(s) \leq G_{M,\lambda}(t,s), \ \text{for all } (t,s) \in [\delta, 1] \times I. \tag{9}$$

Corollary

Let $G_{M,\lambda}(t,s)$ be Green's function related to problem (4)–(5) *given by expression* (8)*. Then if $M \in [-m_0^4, m_1^4)$ and $\lambda \in (0, 1/C_M)$ we have that for all $\delta \in (0,1)$ there exists $\gamma(\delta) \in (0,1)$ for which the following inequality is fulfilled:*

$$\gamma \frac{\lambda}{\lambda\, C_M - 1} z_M(s) \leq G_{M,\lambda}(t,s), \quad \text{for all } (t,s) \in [\delta, 1] \times I. \tag{9}$$

Corollary

Let $G_{M,\lambda}(t,s)$ be Green's function related to problem (4)–(5) given by expression (8). Then if $M \in [-m_0^4, m_1^4)$ and $\lambda \in (0, 1/C_M)$ we have that for all $\delta \in (0,1)$ there exists $\gamma(\delta) \in (0,1)$ for which the following inequality is fulfilled:

$$\gamma \frac{\lambda}{\lambda\, C_M - 1} z_M(s) \le G_{M,\lambda}(t,s), \text{ for all } (t,s) \in [\delta, 1] \times I. \qquad (9)$$

For any pair δ, γ satisfying (9) we introduce the following cone:

$$K_\gamma^\delta = \{u \in C(I),\ u(t) \ge 0 \quad \text{for all } t \in I, \text{ and } \min_{t\in[\delta,1]} u(t) \ge \frac{\gamma}{R} \|u\|\}.$$

LEMMA

Assume that f satisfies condition (H_0), then, $u \in C(I)$ is a solution of (4)-(5) if and only if u is a fixed point of operator

$$T_{M,\lambda}(u(t)) = \int_0^1 G_{M,\lambda}(t,s)\, f(s,u(s))\, ds$$

LEMMA

Assume that f satisfies condition (H_0), *then,* $u \in C(I)$ *is a solution of* (4)-(5) *if and only if u is a fixed point of operator*

$$T_{M,\lambda}(u(t)) = \int_0^1 G_{M,\lambda}(t,s)\, f(s,u(s))\, ds$$

LEMMA

Assuming condition (H_0), *we have that* $T(K_\gamma^\delta) \subset K_\gamma^\delta$ *and it is completely continuous.*

In order to deduce the existence and multiplicity of solutions of the nonlinear problem (4) – (5), we introduce the following constants for any $M \in [-m_0^4, m_1^4)$ (in case of $M = 0$ it must be considered the limit when M goes to zero):

$$\Lambda_1 = \frac{1}{\max\limits_{t \in I} \displaystyle\int_\delta^1 G_{M,\lambda}(t,s)\, ds}.$$

$$\Lambda_2 = \frac{1}{\|\frac{\lambda}{\lambda\, C_M - 1} z_M\|_1} = \frac{(1 - \lambda\, C_M) M}{\lambda\,(z_M'''(1) - z_M'''(0) - 1)}.$$

THEOREM

Let $m \in \mathbb{N} \cup +\infty$ *and* $\{r_k\}_{k=1}^m$ *and* $\{R_k\}_{k=1}^m$ *be such that*

$$r_{k+1} < R_{k+1} < r_k < R_k, \quad k = 1, 2, 3, \ldots, m-1.$$

Furthermore assume that f *satisfies condition* (H_0) *and for each natural number* k *there are* $0 < A < \Lambda_2$ *and* $B > \Lambda_1$ *such that*

Theorem

Let $m \in \mathbb{N} \cup +\infty$ *and* $\{r_k\}_{k=1}^m$ *and* $\{R_k\}_{k=1}^m$ *be such that*

$$r_{k+1} < R_{k+1} < r_k < R_k, \quad k = 1, 2, 3, \ldots, m-1.$$

Furthermore assume that f *satisfies condition* (H_0) *and for each natural number* k *there are* $0 < A < \Lambda_2$ *and* $B > \Lambda_1$ *such that*

(C_1) $f(t,u) \geq B r_k$, *for all* $\gamma\, r_k / R \leq u \leq r_k$, *and* $t \in [\delta, 1]$.

Theorem

Let $m \in \mathbb{N} \cup +\infty$ *and* $\{r_k\}_{k=1}^m$ *and* $\{R_k\}_{k=1}^m$ *be such that*

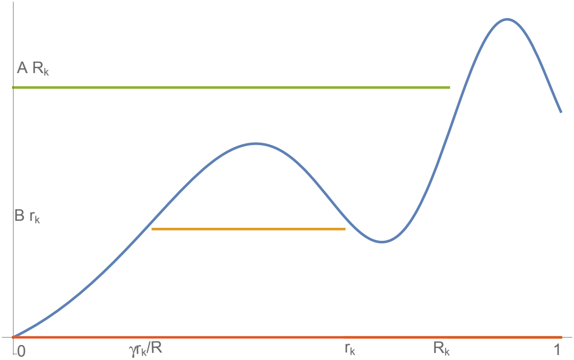
$$r_{k+1} < R_{k+1} < r_k < R_k, \quad k = 1, 2, 3, \ldots, m-1.$$

Furthermore assume that f *satisfies condition* (H_0) *and for each natural number* k *there are* $0 < A < \Lambda_2$ *and* $B > \Lambda_1$ *such that*

(C_1) $f(t,u) \geq B r_k$, *for all* $\gamma\, r_k/R \leq u \leq r_k$, *and* $t \in [\delta, 1]$.

(C_2) $f(t,u) \leq A R_k$, *for all* $0 \leq u \leq R_k$ *and* $t \in [0,1]$,

Multiplicity of Solutions



THEOREM

Then the boundary value problem (4)–(5) *has* $2\,m-1$ *positive solutions* $\{u_k\}_{k=1}^{m}$ *and* $\{v_k\}_{k=1}^{m-1}$ *such that*

$$r_k < \|u_k\| < R_k, \; k = 1, 2, 3, \dots, m,$$

$$R_{k+1} < \|v_k\| < r_k, \; k = 1, 2, 3, \dots, m-1.$$

Theorem

Then the boundary value problem (4)–(5) *has* $2\,m-1$ *positive solutions* $\{u_k\}_{k=1}^{m}$ *and* $\{v_k\}_{k=1}^{m-1}$ *such that*

$$r_k < \|u_k\| < R_k,\ k = 1, 2, 3, \ldots, m,$$

$$R_{k+1} < \|v_k\| < r_k,\ k = 1, 2, 3, \ldots, m-1.$$

Theorem

Then the boundary value problem (4)–(5) *has* $2\,m-1$ *positive solutions* $\{u_k\}_{k=1}^{m}$ *and* $\{v_k\}_{k=1}^{m-1}$ *such that*

$$r_k < \|u_k\| < R_k, \; k = 1, 2, 3, \dots, m,$$

$$R_{k+1} < \|v_k\| < r_k, \; k = 1, 2, 3, \dots, m-1.$$

The proof follows from the iterated application of the Krasnoselskii's Fixed Point Theorem.

$$\Omega_{1,k} = \{u \in K_\gamma^\delta : \|u\| < r_k\}, \; k = 1, 2, \dots, m,$$

and

$$\Omega_{2,k} = \{u \in K_\gamma^\delta : \|u\| < R_k\}, \; k = 1, 2, \dots, m.$$

To deduce other type of multiplicity results, we introduce the following notations:

$$\Lambda_3 = \left\| \frac{M\,\lambda}{\lambda\,C_M - 1} z_M \right\|_1 = \frac{\lambda(z_M'''(1) - z_M'''(0) - 1)}{(1 - \lambda\,C_M)M},$$

and

$$\Lambda_4 = \min_{t \in [\delta,1]} \int_\delta^1 G_{M,\lambda}(t,s)\,ds.$$

THEOREM

Choose $0 < \gamma < 1/R$ and let a, b, c in $\mathbb{R}$ be such that $0 < a < b < \frac{b}{\gamma R} \leq c$. Assume that $B > \Lambda_3$ and $0 < A < \Lambda_4$, condition (H_0) holds and the following properties are fulfilled:

THEOREM

Choose $0 < \gamma < 1/R$ *and let* a, b, c *in* $\mathbb{R}$ *be such that* $0 < a < b < \frac{b}{\gamma R} \leq c$. *Assume that* $B > \Lambda_3$ *and* $0 < A < \Lambda_4$, *condition* (H_0) *holds and the following properties are fulfilled:*

(H_1) *For all* $u \in [0, c]$, *we have* $f(t, u) \leq \dfrac{c}{B}$, $t \in [0, 1]$.

Theorem

Choose $0 < \gamma < 1/R$ and let a, b, c in $\mathbb{R}$ be such that $0 < a < b < \frac{b}{\gamma R} \leq c$. Assume that $B > \Lambda_3$ and $0 < A < \Lambda_4$, condition (H_0) holds and the following properties are fulfilled:

(H_1) *For all $u \in [0, c]$, we have $f(t, u) \leq \dfrac{c}{B}, \quad t \in [0, 1]$.*

(H_2) *For all $u \in [0, a]$, we have $f(t, u) < \dfrac{a}{B}, \quad t \in [0, 1]$.*

THEOREM

Choose $0 < \gamma < 1/R$ *and let* a, b, c *in* $\mathbb{R}$ *be such that* $0 < a < b < \frac{b}{\gamma R} \leq c$. *Assume that* $B > \Lambda_3$ *and* $0 < A < \Lambda_4$, *condition* (H_0) *holds and the following properties are fulfilled:*

(H_1) *For all* $u \in [0, c]$, *we have* $f(t, u) \leq \dfrac{c}{B}, \quad t \in [0, 1]$.

(H_2) *For all* $u \in [0, a]$, *we have* $f(t, u) < \dfrac{a}{B}, \quad t \in [0, 1]$.

(H_3) *For all* $u \in \left[b, \frac{b}{\gamma R}\right]$ *we have* $f(t, u) \geq \dfrac{b}{A}, \quad t \in [\delta, 1]$.

THEOREM

Choose $0 < \gamma < 1/R$ *and let* a, b, c *in* $\mathbb{R}$ *be such that* $0 < a < b < \frac{b}{\gamma R} \leq c$. *Assume that* $B > \Lambda_3$ *and* $0 < A < \Lambda_4$, *condition* (H_0) *holds and the following properties are fulfilled:*

(H_1) *For all* $u \in [0, c]$, *we have* $f(t, u) \leq \dfrac{c}{B}$, $t \in [0, 1]$.

(H_2) *For all* $u \in [0, a]$, *we have* $f(t, u) < \dfrac{a}{B}$, $t \in [0, 1]$.

(H_3) *For all* $u \in \left[b, \frac{b}{\gamma R}\right]$ *we have* $f(t, u) \geq \dfrac{b}{A}$, $t \in [\delta, 1]$.

Theorem

Choose $0 < \gamma < 1/R$ *and let* a, b, c *in* $\mathbb{R}$ *be such that* $0 < a < b < \frac{b}{\gamma R} \leq c$. *Assume that* $B > \Lambda_3$ *and* $0 < A < \Lambda_4$, *condition* (H_0) *holds and the following properties are fulfilled:*

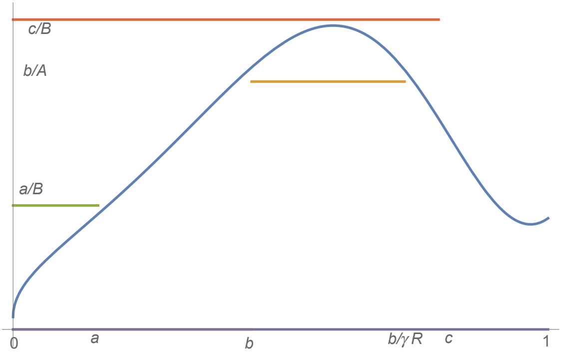
(H_1) *For all* $u \in [0, c]$, *we have* $f(t, u) \leq \dfrac{c}{B}, \; t \in [0, 1]$.

(H_2) *For all* $u \in [0, a]$, *we have* $f(t, u) < \dfrac{a}{B}, \; t \in [0, 1]$.

(H_3) *For all* $u \in \left[b, \frac{b}{\gamma R}\right]$ *we have* $f(t, u) \geq \dfrac{b}{A}, \; t \in [\delta, 1]$.

Then the boundary value problem (4)–(5) *has at least two positive solutions* u_1 *and* u_2 *in* $\overline{P_c} = \{u \in K_\gamma^\delta, \|u\| \leq c\}$, *such that* $\min_{t\in[\delta,1]}\{u_1(t)\} > b$ *and* $\|u_2\| > a$ *with* $\min_{t\in[\delta,1]}\{u_2(t)\} < b$.

Multiplicity of Solutions



The proof follows from the Leggett-Williams fixed point theorem

The proof follows from the Leggett-Williams fixed point theorem

Remark

In case of f satisfies the following condition

The proof follows from the Leggett-Williams fixed point theorem

Remark

In case of f satisfies the following condition

(H_4) $f(t,0) \not\equiv 0$ *on* I,

The proof follows from the Leggett-Williams fixed point theorem

REMARK

In case of f satisfies the following condition

(H_4) $f(t,0) \not\equiv 0$ *on* I,

The proof follows from the Leggett-Williams fixed point theorem

Remark

In case of f satisfies the following condition

(H_4) $f(t,0) \not\equiv 0$ *on* I,

we have three positive solutions of problem (4)–(5).

The proof follows from the Leggett-Williams fixed point theorem

Remark

In case of f satisfies the following condition

(H_4) $f(t,0) \not\equiv 0$ *on* I,

we have three positive solutions of problem (4)–(5).

It is possible to improve this result to deduce m different positive solutions.

In the sequel, we will obtain the different bounds and results for the particular case when $M = 0$. That is, we want to prove the existence of multiple positive solutions of the problem:

$$L_0 u(t) = u^{(4)}(t) = -f(t, u(t)), \quad t \in I \tag{10}$$

subject to the boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \; u(1) = \lambda \int_0^1 u(s)\, ds. \tag{4}$$

with $0 < \lambda < 4$.

$$G_0(t,s)=\begin{cases}\frac{1}{6}\left(-(-1+s)^3t^3-(-s+t)^3\right)+\frac{(1-s)^3st^3\lambda}{6(4-\lambda)} & \text{if } 0<s<t<1\\ -\frac{1}{6}(-1+s)^3t^3+\frac{(1-s)^3st^3\lambda}{6(4-\lambda)} & \text{if } 0\leq t\leq s\leq 1\end{cases}$$

$$G_0(t,s) = \begin{cases} \frac{1}{6}\left(-(-1+s)^3 t^3 - (-s+t)^3\right) + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} & \text{if } 0 < s < t < 1 \\ -\frac{1}{6}(-1+s)^3 t^3 + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} & \text{if } 0 \leq t \leq s \leq 1 \end{cases}$$

$$h(t) = \begin{cases} \frac{4t^3}{\lambda} & \text{if } t \in \left[0, \frac{3}{4}\right] \\ \frac{t^2(-3(-4+\lambda)+4t(-3+\lambda))}{\lambda} & \text{if } t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Example

Choosing $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $A \in (0, \Lambda_2)$ and $B \in (\Lambda_1, +\infty)$ such that $\alpha > \max\left(10, \sqrt{\frac{B}{A}}\right)$, $R_k = \alpha^{-4k}$ and $r_k = \alpha^{-(4k+2)}$ for $k \in \{1, 2, \dots, m-1\}$.
We have that $r_{k+1} < R_{k+1} < r_k < R_k$, $[10^{-2} r_k, \frac{\gamma}{R} r_k] \subset [R_{k+1}, r_k]$, $R_{k+1} < r_k < B r_k$ and $B r_k < A R_k$.

Example

Let f a function defined as follows:

$$f(t,u)=\begin{cases} R_{k+1} & \text{if } u\in[0,R_{k+1}]\\ \dfrac{Br_k-R_{k+1}}{10^{-2}r_k-R_{k+1}}(u-R_{k+1})+R_{k+1} & \text{if } u\in[R_{k+1},10^{-2}r_k]\\ Br_k & \text{if } u\in[10^{-2}r_k,r_k]\\ \dfrac{Br_k-AR_k}{r_k-R_k}(u-R_k)+AR_k & \text{if } u\in[r_k,R_k]\\ AR_k & \text{if } u\in[R_k,+\infty) \end{cases}$$

We have that boundary value problem (10)–(4) has $2\,m-1$ positive solutions $\{u_k\}_{k=1}^m$ such that $\alpha^{-(4k+2)}\le\|u_k\|\le\alpha^{-4k}$ for each $k=1,2,...,m$ and $\{v_k\}_{k=1}^m$ such that $\alpha^{-(4(k+1))}\le\|v_k\|\le\alpha^{-(4k+2)}$ for each $k=1,2,...,m-1$.

Example

We consider the boundary value problem (10) with

$$f(t,u) = \begin{cases} \dfrac{\left[\left(t-\frac{1}{2}\right)^2+1\right]}{10} + 3334u^3 & \text{if } 0 \le u \le 1 \\ 3333 + \dfrac{\left[\left(t-\frac{1}{2}\right)^2+1\right]}{10} + u & \text{if } u > 1\,. \end{cases}$$

The boundary value problem (10)–(4) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_1\| < \frac{1}{30}$, $\min_{t\in[1/4,1]}(u_2(t)) > 1$ and $\|u_3\| > \frac{1}{30}$ with $\min_{t\in[1/4,1]}(u_3(t)) < 1$.

Example

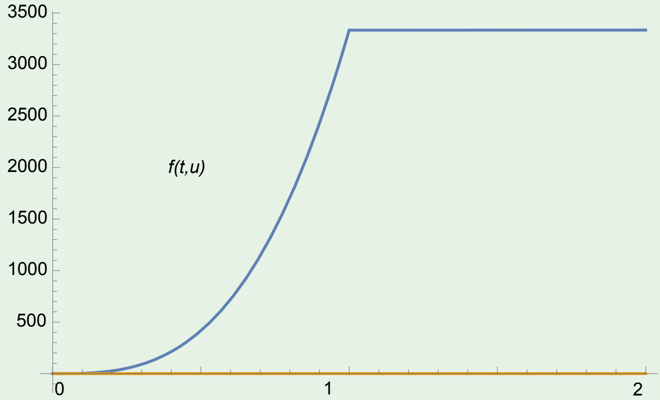


Figure: Graph of $f(1/2, u)$

Example

The boundary value problem (10)–(4) has at least three positive solutions u_1, u_2, u_3 satisfying $\|u_1\| < \frac{1}{30}$, $\min_{t\in[1/4,1]}(u_2(t)) > 1$ and $\|u_3\| > \frac{1}{30}$ with $\min_{t\in[1/4,1]}(u_3(t)) < 1$.

THANKS FOR YOUR ATTENTION