Semigroups of non-Lebesgue measurable sets generated by Vitali selectors of the real line

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Outline

Vitali selectors of the real line (1)

In 1905, the Italian Mathematician G.Vitali introduced the following construction [1],[2].

- Let Q be a countable dense subgroup of the additive topological group $(\mathbb{R}, +)$ of real numbers.
- Define an equivalence relation $\mathcal{R}$ on $\mathbb{R}$ as follows:

$$\text{for } x, y \in \mathbb{R}, \text{ let } x\mathcal{R}y \text{ iff } y - x \in Q.$$

- Let $\mathbb{R}/Q = \{E_\alpha(Q) : \alpha \in \Gamma\}$ denotes the set of all equivalence classes, where Γ is some indexing set.
- Hence we have the following decomposition of $\mathbb{R}$:

$$\mathbb{R} = \bigcup\{E_\alpha(Q) : \alpha \in \Gamma\}$$

Vitali selectors of the real line (2)

Definition 1.1 ([1], [2]).

A Vitali selector of $\mathbb{R}$ related to Q is any subset V of $\mathbb{R}$ satisfying the condition:

$$\mathrm{Card}(\mathrm{V} \cap \mathrm{E}_\alpha(\mathrm{Q})) = 1 \text{ for each } \alpha \in \Gamma$$

Definition 1.2 ([3]).

A subset B of $\mathbb{R}$ is said to have the Baire property in $\mathbb{R}$ if B can be represented as a symmetric difference $\mathrm{B} = \mathrm{O}\Delta\mathrm{M}$, where O is open in $\mathbb{R}$ and M is a first category set on $\mathbb{R}$, i.e. $\mathrm{M} = \bigcup_{i=1}^{\infty} \mathrm{M}_i$ with $\mathrm{Int}\,\mathrm{Cl}(\mathrm{M}_i) = \emptyset$ for each $i = 1, 2, \cdots$.

The family $\mathcal{B}_\mathrm{P}$ of sets having the Baire property in $\mathbb{R}$ is a σ-algebra, containing the collections $\mathcal{B}_\mathrm{O}$ of all Borel sets as well as the collection $\mathcal{M}$ of all first category sets.

Vitali selectors of the real line (3)

If $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ then we write $A + t$ for the set $\{a + t : a \in A\}$.

Proposition 1.1 ([1], [2]).

Let V be a Vitali selector of $\mathbb{R}$. If $q_1, q_2 \in Q$ and $q_1 \neq q_2$ then $(V + q_1) \cap (V + q_2) = \emptyset$, and hence we have another decomposition of $\mathbb{R}$:

$$\mathbb{R} = \bigcup \{V + q : q \in Q\}$$

Theorem 1.3 ([1], [2]).

(1) Each Vitali selector of $\mathbb{R}$ is not measurable in the Lebesgue sense and it does not possess the Baire property.

(2) If V is a Vitali selector related to a countable dense subgroup Q of $(\mathbb{R}, +)$, then $V + t$ is also a Vitali selector related to Q, for any $t \in \mathbb{R}$.

Vitali selectors of the real line (4)

Theorem 1.4 ([4], [5]).

If $\{V_\alpha : 1 \le \alpha \le n\}$ is a non-empty finite family of Vitali selectors related to a countable dense subgroup Q of $(\mathbb{R}, +)$, then the union $\bigcup\{V_\alpha : 1 \le \alpha \le n\}$ is not measurable in the Lebesgue sense, and it does not possess the Baire property.

If Q is a countable dense subgroup of $(\mathbb{R}, +)$, let $\mathcal{V}(Q)$ be the family of all Vitali selectors associated with Q.

It follows from Theorem 1.4 that each finite union of elements of $\mathcal{V}(Q)$ is not measurable in the Lebesgue sense and does not possess the Baire property.

Let us remark that there exist other types of subsets of $\mathbb{R}$ which are not measurable in the Lebesgue sense, and without the Baire property.

Algebraic structures in the families $\mathcal{L}$ and $\mathcal{B}_{\mathrm{P}}$ (1)

- Let $\mathcal{L}$ be the family of all measurable subsets of $\mathbb{R}$. The family $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ can be decomposed in two different ways:

$$\mathbb{R} = \mathcal{B}_{\mathrm{P}} \cup \mathcal{B}_{\mathrm{P}}^{\mathrm{c}} = \mathcal{L} \cup \mathcal{L}^{\mathrm{c}},$$

  where $\mathcal{L}^{\mathrm{c}} = \mathcal{P}(\mathbb{R}) \setminus \mathcal{L}$ and $\mathcal{B}_{\mathrm{P}}^{\mathrm{c}} = \mathcal{P}(\mathbb{R}) \setminus \mathcal{B}_{\mathrm{P}}$.

- Note that $\mathcal{B}_{\mathrm{P}} \cap \mathcal{L} \neq \emptyset, \mathcal{B}_{\mathrm{P}} \cap \mathcal{L}^{\mathrm{c}} \neq \emptyset, \mathcal{L} \cap \mathcal{B}_{\mathrm{P}}^{\mathrm{c}} \neq \emptyset$ and $\mathcal{B}_{\mathrm{P}}^{\mathrm{c}} \cap \mathcal{L}^{\mathrm{c}} \neq \emptyset$.

The algebraic structure of $\mathcal{L}$ and $\mathcal{B}_{\mathrm{P}}$, in the set-theoretic point of view, is well known.

- The families $\mathcal{L}$ and $\mathcal{B}_{\mathrm{P}}$ are σ-algebras of sets, and each contains the collection $\mathcal{B}_{\mathrm{O}}$ of all Borel sets on $\mathbb{R}$.
- The family $\mathcal{L}$ contains the collection $\mathcal{N}_0$ of all sets having the Lebesgue measure zero, while the family $\mathcal{B}_{\mathrm{P}}$ contains the collection $\mathcal{M}$.

Algebraic structures in the families $\mathcal{L}$ and $\mathcal{B}_{\mathrm{P}}$ (2)

Definition 2.1 ([7]).

Let $\mathcal{F}$ be a subfamily of $\mathcal{P}(\mathbb{R})$ and let $\Psi(\mathbb{R})$ be a group of homeomorphisms of $\mathbb{R}$ onto itself. The family $\mathcal{F}$ is said to be invariant under the action of $\Psi(\mathbb{R})$, if for each $A \in \mathcal{F}$ and for each $h \in \Psi(\mathbb{R})$, we have $h(A) \in \mathcal{F}$.

- The family $\mathcal{L}$ is invariant under the action of the group $\mathcal{T}(\mathbb{R})$ of all translations of $\mathbb{R}$.
- However, as it can be observed in [8] and [9], there are homeomorphisms of $\mathbb{R}$ onto itself which do not preserve the elements of $\mathcal{L}$.
- The family $\mathcal{B}_{\mathrm{P}}$ is invariant under the action of the group $\mathcal{H}(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$ onto itself.

Algebraic structures in the families $\mathcal{L}^{\mathrm{c}}$ and $\mathcal{B}^{\mathrm{c}}_{\mathrm{P}}$ (1)

- But the families $\mathcal{L}^{\mathrm{c}}$ and $\mathcal{B}^{\mathrm{c}}_{\mathrm{P}}$ do not have a well-defined structure in the set-theoretic point of view.

- The union (resp. intersection, difference) of two elements in $\mathcal{L}^{\mathrm{c}}$ (resp. $\mathcal{B}^{\mathrm{c}}_{\mathrm{P}}$) can be inside or outside of $\mathcal{L}^{\mathrm{c}}$ (resp. $\mathcal{B}^{\mathrm{c}}_{\mathrm{P}}$).

- The family $\mathcal{L}^{\mathrm{c}}$ is invariant under the action of $\mathcal{T}(\mathbb{R})$, but there are homeomorphisms of $\mathbb{R}$ onto itself which do not preserve the elements of $\mathcal{L}^{\mathrm{c}}$.
- The family $\mathcal{B}^{\mathrm{c}}_{\mathrm{P}}$ is invariant under the action of $\mathcal{H}(\mathbb{R})$.

Semigroups of sets (1)

Problem 3.1 ([5]).

Could we find in $\mathcal{L}^c$ (resp. $\mathcal{B}^c_P$) subfamilies of $\mathcal{P}(\mathbb{R})$ which contain $\mathcal{V}(Q)$, and have some algebraic structures in the set-theoretic point of view?

To suggest some answers to this problem, we need to introduce some technical concepts.

Definition 3.2.

Let $\mathcal{S}$ be a non-empty collection of elements of $\mathcal{P}(\mathbb{R})$. Then $\mathcal{S}$ is said to be a semigroup of sets if for each pair of elements $A, B \in \mathcal{S}$ we have $A \cup B \in \mathcal{S}$.

Example 3.3 ([6]).

Let $\mathcal{A}$ be a family of subsets of $\mathbb{R}$. Set $\mathcal{S}(\mathcal{A}) = \{\bigcup_{i=1}^{n} A_i : A_i \in \mathcal{A}, n \in \mathbb{N}\}$. The family $\mathcal{S}(\mathcal{A})$ is called the semigroup of sets generated by $\mathcal{A}$. Note that $\mathcal{A} \subseteq \mathcal{S}(\mathcal{A})$.

Semigroups of sets (2)

Definition 3.4.

A non-empty collection $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is called an ideal of sets on $\mathbb{R}$ if it satisfies the following conditions:

(i) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

(ii) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.

If an ideal of sets is closed under countable unions then it is said to be a σ-ideal of sets.

- ♣ The collection $\mathcal{N}_0$ of all subsets of $\mathbb{R}$ having the Lebesgue measure zero is σ-ideal of sets.
- ♣ The collection $\mathcal{M}$ of all first category subsets of $\mathbb{R}$ is a σ-ideal of sets.

Semigroups of sets (3)

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of $\mathbb{R}$. Define a new family of sets on $\mathbb{R}$ by setting

$$\mathcal{A} * \mathcal{B} := \{(A \setminus B_1) \cup B_2 : A \in \mathcal{A}, B_1 \in \mathcal{B}, B_2 \in \mathcal{B}\}$$

Proposition 3.1 ([6]).

Let $\mathcal{S}$ be a semigroup of sets on $\mathbb{R}$ and let $\mathcal{I}$ be an ideal of sets on $\mathbb{R}$. Then the families $\mathcal{I} * \mathcal{S}$ and $\mathcal{S} * \mathcal{I}$ are semigroups of sets on $\mathbb{R}$ such that $\mathcal{S} \subseteq \mathcal{I} * \mathcal{S} \subseteq \mathcal{S} * \mathcal{I}$. Moreover, $\mathcal{I} * (\mathcal{I} * \mathcal{S}) = \mathcal{I} * \mathcal{S}$ and $(\mathcal{S} * \mathcal{I}) * \mathcal{I} = \mathcal{S} * \mathcal{I}$.

Let $\mathcal{C}$ be the family of all countable dense subgroups of $(\mathbb{R}, +)$, and let

$$\mathcal{V}_1(Q) = \left\{ \bigcup_{i=1}^{n} V_i : V_i \in \mathcal{V}(Q), n \in \mathbb{N} \right\}$$

be the semigroup generated by the family $\mathcal{V}(Q)$ for some $Q \in \mathcal{C}$.
Note that $\mathcal{V}(Q) \subsetneq \mathcal{V}_1(Q)$ for each $Q \in \mathcal{C}$.

Main results (1)

Theorem 4.1 ([4]).

If $\{V_\alpha : 1 \leq \alpha \leq n\}$ is a non-empty finite family of Vitali selectors of $\mathbb{R}$ related to an element $Q \in \mathcal{C}$, then the union $\bigcup\{V_\alpha : 1 \leq \alpha \leq n\}$ is not measurable in the Lebesgue sense, and it does not possess the Baire property.

- It follows from Theorem 4.1 that the semigroup $\mathcal{V}_1(Q)$ consists of sets which are not measurable in the Lebesgue sense, and without the Baire property in $\mathbb{R}$, for each $Q \in \mathcal{C}$.

Theorem 4.2 ([6]).

For each $Q \in \mathcal{C}$, the families $\mathcal{V}_1(Q), \mathcal{N}_0 * \mathcal{V}_1(Q)$ and $\mathcal{V}_1(Q) * \mathcal{N}_0$ are semigroups of sets on $\mathbb{R}$ such that $\mathcal{V}_1(Q) \subsetneq \mathcal{N}_0 * \mathcal{V}_1(Q) \subsetneq \mathcal{V}_1(Q) * \mathcal{N}_0$, for which elements are not Lebesgue measurable, and they are invariant under the action of the group $\mathcal{T}(\mathbb{R})$.

Theorem 4.2 can be also formulated in the case of the Baire property, where $\mathcal{N}_0$ is replaced by $\mathcal{M}$, and "not Lebesgue measurable" is replaced by "without the Baire property".

Main results (2)

- Theorem 4.1 was proved for a finite union of Vitali selectors related to the same countable dense subgroup Q of $(\mathbb{R}, +)$.
- Namely, Theorem 4.1 shows that each element of the family $\mathcal{V}_1(Q)$ is not measurable in the Lebesgue sense, for any $Q \in \mathcal{C}$.

The following statement generalizes the results of Theorem 4.1.

Theorem 4.3.

Let $U = \bigcup_{i=1}^{n} V_i$ be a finite union of Vitali selectors of $\mathbb{R}$, where $V_i \in \mathcal{V}(Q_i)$ and each Q_i is a countable dense subgroup of $(\mathbb{R}, +)$ for $i = 1, 2, \cdots, n$. Then the set U is not measurable in the Lebesgue sense.

Theorem 4.3 can be also formulated in the case of the Baire property, where ”not measurable in the Lebesgue sense” is replaced by ”without the Baire property” [6].

Main results (3)

- Let $\mathcal{V}^{\mathrm{sup}} = \{V : V \in \mathcal{V}(Q), Q \in \mathcal{C}\}$ be the family of all Vitali selectors of $\mathbb{R}$. One can define the semigroup $\mathcal{V}_1^{\mathrm{sup}} = \{\bigcup_{i=1}^{n} V_i : V_i \in \mathcal{V}^{\mathrm{sup}}\}$ generated by $\mathcal{V}^{\mathrm{sup}}$.
- It is clear that $\mathcal{V}(Q) \subsetneq \mathcal{V}^{\mathrm{sup}}$ and $\mathcal{V}_1(Q) \subsetneq \mathcal{V}_1^{\mathrm{sup}}$ for each $Q \in \mathcal{C}$.

Theorem 4.4.

The families $\mathcal{N}_0 * \mathcal{V}_1^{\mathrm{sup}}$ and $\mathcal{V}_1^{\mathrm{sup}} * \mathcal{N}_0$ are semigroups of sets, for which elements are not measurable in the Lebesgue sense, such that $\mathcal{V}_1^{\mathrm{sup}} \subsetneq \mathcal{N}_0 * \mathcal{V}_1^{\mathrm{sup}} \subsetneq \mathcal{V}_1^{\mathrm{sup}} * \mathcal{N}_0$, and they are invariant under the action of the group $\mathcal{A}(\mathbb{R})$ of all affine transformations of $\mathbb{R}$ into itself.

Theorem 4.4 can be also formulated in the case of the Baire property, where $\mathcal{N}_0$ is replaced by $\mathcal{M}$, and "not Lebesgue measurable in the Lebesgue sense" is replaced by "without the Baire property" [6].

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