



# Approximation of the two variables Discontinuous Functions by Discontinuous Interpolation Splines using Triangular Elements

**Iuliia Pershyna**

[yuliapershina78@gmail.com](mailto:yuliapershina78@gmail.com)

D.Sc. in Physical and Mathematical Sciences

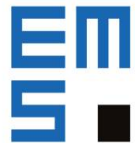
Professor of Department of Information Computer  
Technologies and Mathematics

Ukrainian Engineering Pedagogics Academy,  
Kharkov, Ukraine

# Introduction



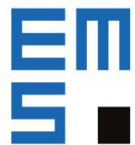
20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



My work is devoted to solving the plane problem of radon computed tomography using the inhomogeneity of the internal structure of a two-dimensional body. For this purpose, it is advisable to use interlination operators, since these operators restore (possibly approximated) functions by their known traces on a given system of lines. They provide a method to construct operators whose integrals from the lines (linear integrals) will be equal to integrals from the most renewable function. I.e. interlination is a mathematical apparatus, naturally related to the problem of restoring the characteristics of objects according to their known projections.



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA

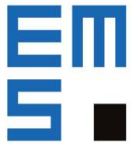


## Formulation of the Problem

Let the function  $f(x, y)$  be a discontinuous function of two variables in the domain  $D$ . We will assume that the domain  $D$  is divided into arbitrary triangles. The sides of triangles do not intersect. The function  $f(x, y)$  has first-order discontinuities at the boundaries between these triangular elements (not necessarily all). The experimental data are unilateral traces of a function along a system of the given lines. Based on these data, let us construct an analytic formula of an unknown function (i.e. the discontinuous interlination operator).

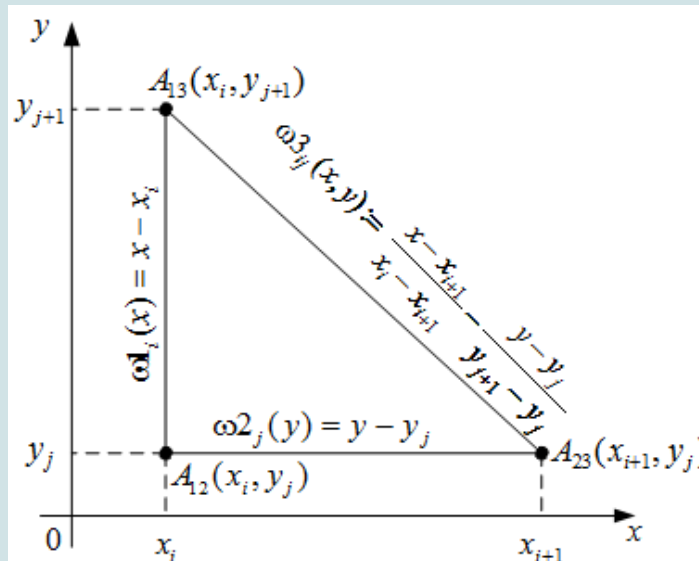


20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



# Approximation of Discontinuous Functions of two variables by Discontinuous interlination splines

Consider a triangular element  $T_{ij}, i = \overline{1, n}, j = \overline{1, m}$



Let's us consider given:

- Function traces  $f(x, y)$  on the straight line  $x = x_i$  (on the right and on the left of the line respectively):

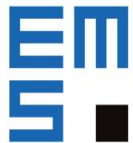
$$\varphi p_i(y) = \lim_{x \rightarrow x_i + 0} f(x, y), \quad \varphi m_i(y) = \lim_{x \rightarrow x_i - 0} f(x, y);$$

$$\varphi p p_{ij} = \varphi p_i(y_j) = \lim_{\substack{x \rightarrow x_i + 0 \\ y \rightarrow y_j + 0}} f(x, y),$$

$$\varphi m p_{ij} = \varphi m_i(y_j) = \lim_{\substack{x \rightarrow x_i - 0 \\ y \rightarrow y_j + 0}} f(x, y)$$



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



- Function traces on the straight line  $y = y_j$  (on the right and on the left of the line respectively).  
Formulas are developed similarly to the traces along the line  $x = x_i$ .

- Function traces on the straight line  $y = \frac{(y_j - y_{j+1})(x - x_i)}{x_{i+1} - x_i} + y_{j+1}$   
(on the right and on the left of the line respectively):

$$\eta m_{ij}(x) = f\left(x, \frac{(y_j - y_{j+1})(x - x_i)}{x_{i+1} - x_i} + y_{j+1} - 0\right),$$

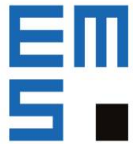
$$\eta p_{ij}(x) = f\left(x, \frac{(y_j - y_{j+1})(x - x_i)}{x_{i+1} - x_i} + y_{j+1} + 0\right), \quad \eta p m_{ij} = \eta m_{ij}(x_i) = f(x_i + 0, y_{j+1} - 0),$$

or

$$\eta m_{ij}(y) = f\left(\frac{(y - y_j)(x_i - x_{i+1})}{y_{j+1} - y_j} + x_{i+1} + 0, y\right), \quad \eta p_{ij}(y) = f\left(\frac{(y - y_j)(x_i - x_{i+1})}{y_{j+1} - y_j} + x_{i+1} - 0, y\right),$$



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



**Theorem 1.** Let the traces of the function  $f(x, y)$  satisfy the conditions

$$\psi p_j(x_i) = \varphi p_i(y_j), \quad \eta m_{ij}(x_i) = \varphi p_i(y_{j+1}), \quad \eta m_{ij}(x_{i+1}) = \psi p_j(x_{i+1}). \quad (1)$$

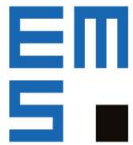
That is, the tracks at the intersection points of the lines coincide. Then operator

$$\begin{aligned} Of(x, y) = & \frac{\omega_{3ij}(x, y)}{\omega_{3ij}(A_{12})} \left( \psi p_j(x) + \varphi p_i(y) - \varphi p_i(y_j) \right) + \frac{\omega_{2j}(y)}{\omega_{2j}(A_{13})} \left( \eta m_{ij}(x) - \right. \\ & \left. - \varphi p_i \left( y_j + \frac{(x - x_{i+1})(y_{j+1} - y_j)}{(x_i - x_{i+1})} \right) + \varphi p_i(y) \right) + \frac{\omega_{1i}(x)}{\omega_{1i}(A_{23})} \times \\ & \times \left( \eta m_{ij} \left( x_{i+1} + \frac{(y - y_j)(x_i - x_{i+1})}{(y_{j+1} - y_j)} \right) - \psi p_j \left( x_{i+1} + \frac{(y - y_j)(x_i - x_{i+1})}{(y_{j+1} - y_j)} \right) + \psi p_j(x) \right) \end{aligned} \quad (2)$$

is the operator of interlination of a function  $f(x, y)$  on  $\partial T_i : Of(x, y)|_{\partial T_i} = f(x, y)|_{\partial T_i}$ .



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



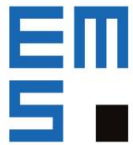
**Theorem 2.** If  $f(x, y)$  is a continuous function together with its partial derivatives up to the second order inclusively inside a triangular element  $T_{ij}$ , then for the residual

$Rf(x, y) = (I - O)f$  equals

$$\begin{aligned}
 Rf(x, y) = & \\
 = & \frac{\omega_{3ij}(x, y)}{\omega_{3ij}(A_{12})} \int_{x_i}^x \int_{y_j}^y f^{(1,1)}(u, v) dudv + \frac{\omega_{2j}(y)}{\omega_{2j}(A_{13})} \int_{x_i}^x \int_{y_j + \frac{(x-x_{i+1})(y_{j+1}-y_j)}{x_i-x_{i+1}}}^y f^{(1,1)}(u, v) dudv + \\
 & + \frac{\omega_{1i}(x)}{\omega_{1i}(A_{23})} \int_{x_{i+1} + \frac{(y-y_j)(x_i-x_{i+1})}{y_{j+1}-y_j}}^x \int_{y_j}^y f^{(1,1)}(u, v) dudv, (x, y) \in T_{ij}.
 \end{aligned}$$



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



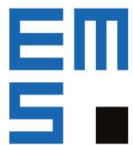
**Theorem 3.** Let  $f(x, y) \in L_{\infty}^{1,1}(T_{ij}), \forall (x, y) \in T_{ij}$ , then there are the following error estimates  $\forall (x, y) \in T_{ij}, :$

$$|Rf(x, y)| \leq \left\| f^{(1,1)}(x, y) \right\|_{L_{\infty}(T_{ij})} \left| (x - x_i)(y - y_j) \left( \frac{x - x_{i+1}}{x_i - x_{i+1}} - \frac{y - y_j}{y_{j+1} - y_j} \right) \right|,$$

where  $L_{\infty}^{1,1}(T_{ij}) = \lim_{p \rightarrow \infty} L_p(T_{ij}) = \sup \text{vrai} |f(x, y)|$  – significant upper bound of the function  $|f(x, y)|$  on  $T_{ij}$ , that is, the smallest of the numbers  $K \geq 0$  for which the inequality  $|f(x, y)| > K$  holds on the set of measure zero.



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA

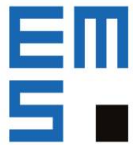


## Approximation of the Discontinuous Two Variables Function by a Discontinuous Interpolation Spline and Triangular Elements with One Curved Side

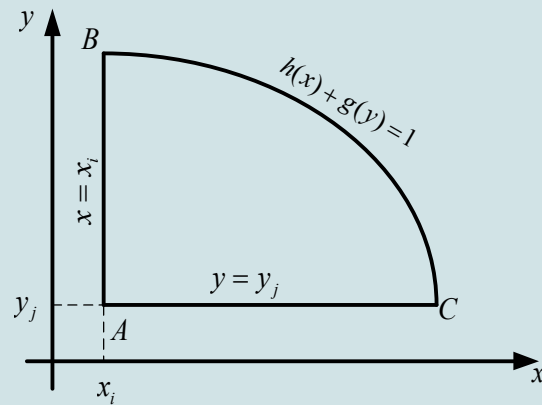
Let a discontinuous function of two variables  $f(x, y)$  defined in the domain  $D$ . We assume that the region  $D$  is divided by lines  $x_0 = 0 < x_1 < x_2 < \dots < x_m = 1$ ,  $y_0 = 0 < y_1 < y_2 < \dots < y_n = 1$ , into rectangular elements, and each rectangle is divided into two right triangles with curvilinear hypotenuse. Triangles do not fit into each other, and the sides of the triangles do not intersect. The function  $f(x, y)$  has discontinuities of the first kind at the boundaries between these right triangles (not necessarily between all). We construct an operator of discontinuous piecewise polynomial interpolation such that in each triangle it is an operator of polynomial interpolation of a function  $f(x, y)$ .



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



Consider a triangular element  $T_{ij}, i = \overline{1, n}, j = \overline{1, m}$

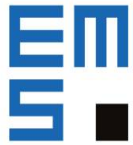


Lets us consider given:

- 1) traces of the function  $f(x, y)$  on the line  $x = x_i$  (to the right and to the left of the line);
- 2) traces of the function  $f(x, y)$  on the line  $y = y_j$  (to the right and to the left of the line);
- 3) traces of a function  $f(x, y)$  on a curved hypotenuse (below and above of the line).



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



**Theorem 5.** If traces of the function  $f(x, y)$  satisfy the conditions

$$\psi p_j(x_i) = \varphi p_i(y_j), \quad \eta m_{ij}(x_i) = \varphi p_i \left( g^{-1}(1 - h(x_i)) \right),$$

$$\eta m_{ij} \left( h^{-1}(1 - g(y_j)) \right) = \psi p_j \left( h^{-1}(1 - g(y_j)) \right),$$

then the operator

$$Lf(x, y) = L_1 f(x, y) + L_2(x, y) - L_{12}(x, y) \quad (3)$$

$$L_1 f(x, y) = h(x) \cdot \eta m_{ij}(y) + g(y) \cdot \eta m_{ij}(x),$$

$$L_2 f(x, y) = \psi p_j(x) + \varphi p_i(y) - \varphi p_i(y_j),$$

$$L_1 L_2 f(x, y) = h(x) \left( \varphi p_i(y) + \mu p_j(y) - \varphi p_i(y_j) \right) +$$

$$g(y) \left( \eta m_{ij}(x_i) + \mu m_i(x) - \psi p_j(x) \right),$$

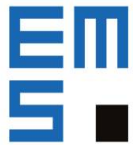
interlined a function  $f(x, y)$  on three sides of a triangle  $T_{ij}$ ,  $i = \overline{1, n}$ ,  $j = \overline{1, m}$ , i.e.

$$L f(x, y) \Big|_{y=y_j} = \psi p_j(x), \quad L f(x, y) \Big|_{x=x_i} = \varphi p_i(y),$$

$$L f(x, y) = f(x, y), \quad \text{if } h(x) + g(y) = 1.$$



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



**Theorem 6.** If  $f(x, y) \in C^{(1,1)}(T_{ij})$ , then for the residual  $Rf(x, y) = (I - L)f(x, y)$  equals

$$Rf(x, y) = (1 - h(x) - g(y)) \int_0^x \int_0^y f^{(1,1)}(u, v) dudv +$$

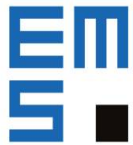
$$+ f(x) \int_x^{h^{-1}(1-g(y))} \int_0^y f^{(1,1)}(u, v) dudv + g(y) \int_0^x \int_y^{g^{-1}(1-h(x))} f^{(1,1)}(u, v) dudv.$$

**Remark.** For arbitrary functions  $f(x, y) = u(x) + v(y)$ , where  $u(x), v(y)$  are arbitrary functions of one variable, equality holds  $Lf(x, y) = f(x, y)$ .

**Comment.** If  $\varphi p_i(y) = \varphi m_i(y)$ ,  $\psi p_j(x) = \psi m_j(x)$ ,  $\eta p_{ij}(x) = \eta m_{ij}(x)$ , then the constructed discontinuous spline of the form (3) is a continuous interlination spline at the boundaries of a triangular element  $T_{ij}$ .



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA

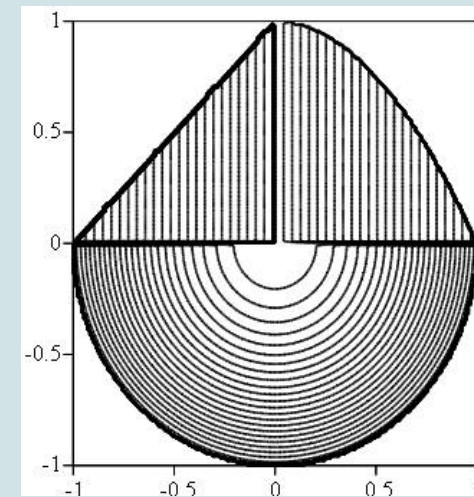


**Example.** Let the function  $f(x,y)$  be defined in the domain  $T = T_1 \cup T_2 \cup T_3 \cup T_4$  as shown in Fig. 4a. The cathetus of these triangular elements are formed by straight lines  $x=0, y=0$ , and the hypotenuses given by an equation of the form  $h(x) + g(y) = 1$ , where they satisfy the conditions  $h(x), g(y)$  and are defined in each triangular element as follows:

$$T_1: h(x) = x^2, g(y) = y, \quad T_2: h(x) = -x, g(y) = y,$$

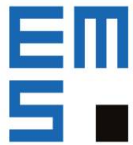
$$T_3: h(x) = x^2, g(y) = y^2, \quad T_4: h(x) = x^2, g(y) = y^2.$$

$$f(x,y) = \begin{cases} 0.5, & 0 < x < 1, 0 < y < 1 - x^2, \\ -x + 1, & -1 < x < 0, 0 < y < 1 + x, \\ x^2 + y^2, & -\sqrt{1 - y^2} < x < \sqrt{1 - y^2}, -1 < y < 0. \end{cases}$$





20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



As initial data we will use the traces of a given function on the lines of triangular elements:

$$T_1: \varphi p(y) = f(+0, y) = 0.5, \quad \psi p(x) = f(y, +0) = 0.5,$$

$$\eta m(x) = f(x, 1 - x^2 - 0) = 0.5 \Rightarrow \eta m(y) = 0.5$$

$$T_2: \varphi m(y) = f(-0, y) = 1, \quad \psi p(x) = f(x, +0) = -x + 1,$$

$$\eta p(y) = f(x, 1 + x + 0) = -x + 1$$

$$T_3: \varphi m(y) = f(-0, y) = y^2, \quad \psi m(x) = f(x, -0) = x^2,$$

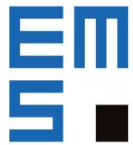
$$\eta p(x) = f(x, -\sqrt{1 - x^2} + 0) = 1$$

$$T_4: \varphi p(y) = f(+0, y) = y^2, \quad \psi m(x) = f(x, -0) = x^2,$$

$$\eta m(x) = f(x, -\sqrt{1 - x^2} - 0) = 1$$



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



We construct the operator  $L1(x, y)$  on  $T_1$ . By the theorem 5 it has a view

$$L1f(x, y) = L_1f(x, y) + L_2(x, y) - L_{12}(x, y)$$

$$L_1f(x, y) = h(x) \cdot \eta m(y) + g(y) \cdot \eta m(x) = x^2 \cdot 0,5 + y \cdot 0,5$$

$$L_2f(x, y) = \psi p(x) + \varphi p(y) - \varphi p(0) = 0,5 + 0,5 - 0,5 = 0,5$$

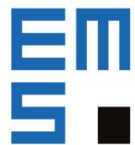
$$\begin{aligned} L_1L_2f(x, y) &= h(x)(\varphi p(y) + \mu p(y) - \varphi p(0)) + g(y)(\eta m(0) + \mu m(x) - \psi p(x)) = \\ &= x^2(0,5 + 0,5 - 0,5) + y(0,5 - 0,5 + 0,5) = 0,5x^2 + 0,5y \end{aligned}$$

$$L1f(x, y) = 0,5x^2 + 0,5y + 0,5 - 0,5x^2 - 0,5y = 0,5$$

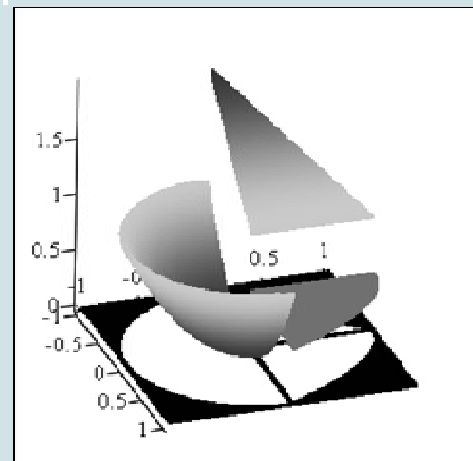
That is, the linear function of the interlination operator restored exactly.



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



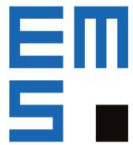
Operators  $L_2, L_3, L_4$  are constructed in a similar way. As a result, we obtain the discontinuous interlinaton operator, which completely coincides with the given function



# Conclusions



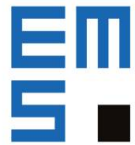
20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



Paper develops a method for approximation of the discontinuous functions of two variables by discontinuous interpolation splines using triangular elements. Experimental data are one-sided traces of a function given along a system of lines (such data are commonly used in remote methods, in particular in tomography). The paper also proposes a method for approximating the discontinuous functions of two variables taking into account triangular elements having one curved side. Proposed methods improve approximation of the discontinuous functions, allowing an application to more complex domains of the definition and avoiding the Gibbs phenomenon.



20 - 26  
JUNE  
2021  
PORTOROŽ  
SLOVENIA



# Thanks for attention