Real Cayley-Dickson algebras: doubly alternative elements and zero divisor graphs

Svetlana Zhilina

Scientific advisor: Alexander Guterman

Lomonosov Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Higher Algebra

June 22, 2021

Orthogonality and zero divisor graphs

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, $Z(\mathcal{A})$ be the set of its zero divisors, $Z_{LR}(\mathcal{A})$ be the set of its two-sided zero divisors. We introduce the following relation graphs of $\mathcal{A}$.

Definition 1

The orthogonality graph $\Gamma_O(\mathcal{A})$:

- the vertex set is $P(Z_{LR}(\mathcal{A})) = \{[a] = \mathbb{F}^* a \mid a \in Z_{LR}(\mathcal{A})\}$,
- $[a]$ and $[b]$ are connected $\Leftrightarrow [a] \neq [b]$, $ab = ba = 0$.

The directed zero divisor graph $\Gamma_Z(\mathcal{A})$:

- the vertex set is $P(Z(\mathcal{A})) = \{[a] = \mathbb{F}^* a \mid a \in Z(\mathcal{A})\}$,
- there is an edge from $[a]$ to $[b]$ $\Leftrightarrow [a] \neq [b]$, $ab = 0$.

The edges of $\Gamma_O(\mathcal{A})$ and $\Gamma_Z(\mathcal{A})$ are well-defined. When speaking of the vertices of these graphs, we will not distinguish between a nonzero element a and a line $[a] = \mathbb{F}a$ passing through it.

Cayley-Dickson process

Definition 2

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ with an involution $a \mapsto \bar{a}$. The algebra $\mathcal{A}\{\gamma\}$ produced by the Cayley-Dickson process, when applied to $\mathcal{A}$ with the parameter $\gamma \in \mathbb{F}$, $\gamma \neq 0$, is defined as the set of ordered pairs of elements of $\mathcal{A}$ with operations

$$\alpha(a, b) = (\alpha a, \alpha b);$$
$$(a, b) + (c, d) = (a + c, b + d);$$
$$(a, b)(c, d) = (ac + \gamma \bar{d} b, da + b\bar{c})$$

and the involution

$$\overline{(a, b)} = (\bar{a}, -b), \qquad a, b, c, d \in \mathcal{A},\ \alpha \in \mathbb{F}.$$

If $\mathcal{A}$ is unital and the involution on $\mathcal{A}$ is regular, that is, $a + \bar{a} \in \mathbb{F}1_{\mathcal{A}}$ and $a\bar{a} = \bar{a}a \in \mathbb{F}1_{\mathcal{A}}$ for all $a \in \mathcal{A}$, then the involution on $\mathcal{A}\{\gamma\}$ is also regular.

Real Cayley-Dickson algebras

We now assume that $\mathbb{F} = \mathbb{R}$.

Definition 3

Let $n \in \mathbb{N}_0$. The algebra

$$\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\} = (\dots(\mathbb{R}\{\gamma_0\})\dots)\{\gamma_{n-1}\}$$

is *a real Cayley-Dickson algebra* determined by the parameters $\gamma_0, \dots, \gamma_{n-1} \in \mathbb{R} \setminus \{0\}$.

In other words, $\mathcal{A}_n$ is defined recursively by using the identities $\mathcal{A}_0 = \mathbb{R}$ and $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\} = (\mathcal{A}_{n-1}\{\gamma_0, \dots, \gamma_{n-2}\})\{\gamma_{n-1}\}$. Clearly, $\dim \mathcal{A}_n = 2^n$.

Proposition 4

$Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$.

Examples of real Cayley-Dickson algebras

$\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is isomorphic to $\mathcal{A}_n\{\mathrm{sgn}(\gamma_0), \dots, \mathrm{sgn}(\gamma_{n-1})\}$, so it is sufficient to consider $\gamma_k \in \{\pm 1\}$ only, $k = 0, \dots, n-1$.

- If $\gamma_0 = \dots = \gamma_{n-1} = -1$ then $\mathcal{A}_n = \mathcal{M}_n = \mathcal{M}_{n-1}\{-1\}$ is an algebra *of the main sequence*.

The complex numbers:	$\mathbb{C} \cong \mathcal{M}_1$;
The quaternions:	$\mathbb{H} \cong \mathcal{M}_2$;
The octonions:	$\mathbb{O} \cong \mathcal{M}_3$;
The sedenions:	$\mathbb{S} \cong \mathcal{M}_4$;

- If $\gamma_0 = \dots = \gamma_{n-2} = -1$ and $\gamma_{n-1} = 1$ then $\mathcal{A}_n = \mathcal{H}_n = \mathcal{M}_{n-1}\{1\}$ is *a Cayley-Dickson split-algebra*.

The split-complex numbers:	$\hat{\mathbb{C}} \cong \mathcal{H}_1$;
The split-quaternions:	$\hat{\mathbb{H}} \cong \mathcal{H}_2$;
The split-octonions:	$\hat{\mathbb{O}} \cong \mathcal{H}_3$;
The split-sedenions:	$\hat{\mathbb{S}} \cong \mathcal{H}_4$.

The real part and the norm

For any $a \in \mathcal{A}_n$ we define

- *real part*: $\mathfrak{Re}(a) = \frac{a+\bar{a}}{2}$;
- *imaginary part*: $\mathfrak{Im}(a) = \frac{a-\bar{a}}{2}$;
- *norm*: $n(a) = a\bar{a} = \bar{a}a$.

Proposition 5

We can compute real part and norm of an element $(a, b) \in \mathcal{A}_{n+1}$ inductively by using the following equalities:

$$\begin{aligned} \mathfrak{Re}((a,b)) &= \mathfrak{Re}(a), \\ n((a,b)) &= n(a) - \gamma_n n(b). \end{aligned}$$

Definition 6

An element $(a, b) \in \mathcal{A}_{n+1}$ is said to be

- *pure* if $\mathfrak{Re}(a) = 0$,
- *doubly pure* if $\mathfrak{Re}(a) = \mathfrak{Re}(b) = 0$.

Alternative elements

- The associator of $a, b, c \in \mathcal{A}$ is $[a, b, c] = (ab)c - a(bc)$.
- $\mathcal{A}$ is called *flexible* if $[a, b, a] = 0$ for all $a, b \in \mathcal{A}$.
- If $\mathcal{A}$ is flexible then $[a, b, c] = -[c, b, a]$ for all $a, b, c \in \mathcal{A}$.
- $\mathcal{A}$ is called *alternative* if $[a, a, b] = [b, a, a] = 0$ for all $a, b \in \mathcal{A}$.
- It is well known that $\mathcal{A}_n$ is alternative if and only if $n \leq 3$, however, $\mathcal{A}_n$ is always flexible.

Definition 7 (Moreno, 2006)

Let $a, b \in \mathcal{A}_n$.

- We say that a *alternates* with b if $[a, a, b] = 0$.
- If a alternates with every $b \in \mathcal{A}_n$ then a is *alternative*.
- We say that a *alternates strongly* with b if $[a, a, b] = 0$ and $[b, b, a] = 0$.
- If a alternates strongly with every $b \in \mathcal{A}_n$ then a is *strongly alternative*.

Zero divisors with restrictions on norm and alternativity

We now assume that $a, b \in \mathcal{A}_n$ alternate strongly with $c, d \in \mathcal{A}_n$, $(a, b)(c, d) = 0$ in $\mathcal{A}_{n+1}$.

Lemma 8

Let $n(c) - \chi\gamma_n n(d) = \chi n(c) - \gamma_n n(d) = 0$ *for some* $\chi \in \mathbb{R}$. *Then* $(c, d)(\overline{ac}, -\chi da) = 0$.

Proof.

Indeed, we have

$$\begin{aligned}
(c, d)(\overline{ac}, -\chi da) &= \left(c(\overline{ac}) + \gamma_n(\overline{-\chi da})d, (-\chi da)c + d(ac)\right) = \\
&= \left(c(\bar{c}\bar{a}) - \chi\gamma_n(\bar{a}\bar{d})d, \chi(b\bar{c})c - \gamma_n d(\bar{d}b)\right) = \\
&= \left((c\bar{c})\bar{a} - \chi\gamma_n\bar{a}(\bar{d}d), \chi b(\bar{c}c) - \gamma_n(d\bar{d})b\right) = \\
&= \left((n(c) - \chi\gamma_n n(d))\bar{a}, (\chi n(c) - \gamma_n n(d))b\right) = 0.
\end{aligned}$$

□

The norm condition $(*)$

Remark 9

If $n(c) = n(d) = 0$ *in Lemma 8 then we can take any* $\chi \in \mathbb{R}$*.* *Otherwise, we obtain immediately*

$$\begin{cases} (n(c))^2 = (n(d))^2 \neq 0; \\ \chi = \gamma_n \dfrac{n(c)}{n(d)} = \gamma_n \dfrac{n(d)}{n(c)} = \pm 1. \end{cases} \tag{$*$}$$

Condition $(*)$ is satisfied automatically if $\mathcal{A}_{n+1}$ is an algebra of the main sequence or if $\mathcal{A}_{n+1}$ is a Cayley-Dickson split-algebra, since in this case $n(c) = n(d)$. The values of χ are equal to -1 and 1, respectively. However, condition $(*)$ is not true in general.

Extending condition $(*)$

Lemma 10 (Moreno, 2006)

If x alternates with y in $\mathcal{A}_n$ then $n(xy) = n(yx) = n(x)n(y)$.

Remark 11

Let (c, d) satisfy condition $(*)$ and $(n(a))^2 + (n(b))^2 \neq 0$. Assume without loss of generality that $n(b) \neq 0$. By Lemma 10, we have

$$n(a)n(d) = n(da) = n(-b\bar{c}) = n(b)n(\bar{c}) = n(b)n(c).$$

Then $\gamma_n \frac{n(a)}{n(b)} = \gamma_n \frac{n(c)}{n(d)} = \chi$. Moreover, Lemma 10 implies that

$$\gamma_n \frac{n(\overline{ac})}{n(-\chi da)} = \gamma_n \frac{n(ac)}{n(da)} = \gamma_n \frac{n(a)n(c)}{n(a)n(d)} = \gamma_n \frac{n(c)}{n(d)} = \chi.$$

Hence (a, b) and $(\overline{ac}, -\chi da)$ also satisfy condition $(*)$.

A hexagon in $\Gamma_Z(\mathcal{A}_{n+1})$

Lemma 12

The elements ac, da *alternate strongly with* a, b, c, d.

Hence we may successively apply Lemma 8 to obtain the following corollary.

Corollary 13

Let (a, b) *and* (c, d) *satisfy condition* $(*)$. *Then there exists the following* 6*-cycle in* $\Gamma_Z(\mathcal{A}_{n+1})$ *which we call a hexagon:*

$$(a, b) \to (c, d) \to (\overline{ac}, -\chi da) \to (a, -b) \to \\ \to (c, -d) \to (\overline{ac}, \chi da) \to (a, b).$$

Properties of zero divisors in $\mathcal{M}_{n+1}$

We now consider the case when $\mathcal{A}_{n+1}$ is an algebra of the main sequence, that is, $\mathcal{A}_{n+1} = \mathcal{M}_{n+1}$.

Lemma 14 (Moreno, 1998)

- *Let* $x, y \in \mathcal{M}_{n+1}$*. Then* $xy = 0 \Leftrightarrow yx = 0$*.*
- *If* $x \in Z(\mathcal{M}_{n+1})$ *then* x *is doubly pure.*
- *Let* $x = (x_1, x_2), y = (y_1, y_2), \tilde{y} = (-y_2, y_1) \in \mathcal{M}_{n+1}$*. Then* $xy = 0 \Leftrightarrow x\tilde{y} = 0$*.*

The next corollary follows immediately from this lemma.

Corollary 15

$\Gamma_Z(\mathcal{M}_{n+1})$ *can be obtained from* $\Gamma_O(\mathcal{M}_{n+1})$ *by replacing every undirected edge with a pair of directed edges.*

Zero divisors with alternativity conditions in $\mathcal{M}_{n+1}$

In the case of the algebras of the main sequence Corollary 13 attains the following form.

Corollary 16

Let $a, b \in \mathcal{M}_n$ *alternate strongly with* $c, d \in \mathcal{M}_n$, $(a, b)(c, d) = 0$ *in* $\mathcal{M}_{n+1}$. *Then there exists the following* 6*-cycle in* $\Gamma_O(\mathcal{M}_{n+1})$*:*

$$\begin{aligned}(a, b) \leftrightarrow (c, d) \leftrightarrow (ac, ad) \leftrightarrow (a, -b) \leftrightarrow& \\ \leftrightarrow (c, -d) \leftrightarrow (ac, -ad) \leftrightarrow (a, b).&\end{aligned}$$

By using Lemma 14 this hexagon can be extended to a double hexagon depicted in Figure 1.

A double hexagon in $\Gamma_O(\mathcal{M}_{n+1})$

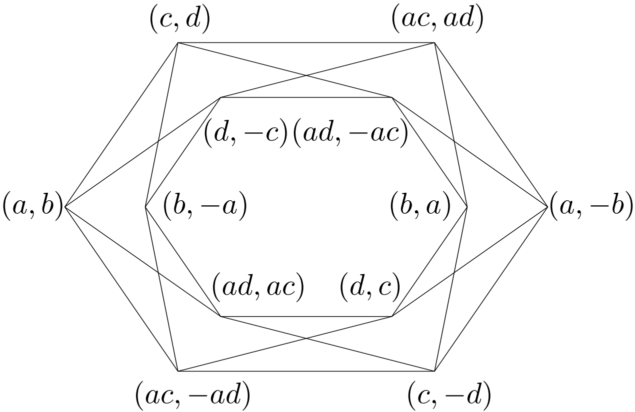

Figure 1: A double hexagon.

Multiplication table of the vertices of a double hexagon

Theorem 17

The multiplication table of the vertices of a double hexagon is given by Table 1, where

$$f_0 = (e_0, 0) = e_0, \qquad \tilde{f}_0 = (0, e_0),$$

$$\begin{aligned}
f_1 &= (a, b), & f_4 &= (ac, -ad), & f_7 &= (0, ab),\\
f_2 &= (c, -d), & f_5 &= (c, d), & f_8 &= (0, dc),\\
f_3 &= (ac, ad), & f_6 &= (a, -b), & f_9 &= (0, (ac)(ad)),\\
\tilde{f}_1 &= (-b, a), & \tilde{f}_4 &= (ad, ac), & \tilde{f}_7 &= (-ab, 0),\\
\tilde{f}_2 &= (d, c), & \tilde{f}_5 &= (-d, c), & \tilde{f}_8 &= (-dc, 0),\\
\tilde{f}_3 &= (-ad, ac), & \tilde{f}_6 &= (b, a), & \tilde{f}_9 &= (-(ac)(ad), 0).
\end{aligned}$$

Multiplication table of the vertices of a double hexagon

$\times$	f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\tilde{f}_0$	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$	$\tilde{f}_4$	$\tilde{f}_5$	$\tilde{f}_6$
f_0	f_0	f_1	f_2	f_3	f_4	f_5	f_6	$\tilde{f}_0$	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$	$\tilde{f}_4$	$\tilde{f}_5$	$\tilde{f}_6$
f_1	f_1	$-2f_0$	$2f_3$	$-2f_2$	0	0	$2f_7$	$\tilde{f}_1$	$-2\tilde{f}_0$	$-2\tilde{f}_3$	$2\tilde{f}_2$	0	0	$-2\tilde{f}_7$
f_2	f_2	$-2f_3$	$-2f_0$	$2f_1$	0	$2f_8$	0	$\tilde{f}_2$	$2\tilde{f}_3$	$-2\tilde{f}_0$	$-2\tilde{f}_1$	0	$-2\tilde{f}_8$	0
f_3	f_3	$2f_2$	$-2f_1$	$-2f_0$	$2f_9$	0	0	$\tilde{f}_3$	$-2\tilde{f}_2$	$2\tilde{f}_1$	$-2\tilde{f}_0$	$-2\tilde{f}_9$	0	0
f_4	f_4	0	0	$-2f_9$	$-2f_0$	$-2f_6$	$2f_5$	$\tilde{f}_4$	0	0	$2\tilde{f}_9$	$-2\tilde{f}_0$	$2\tilde{f}_6$	$-2\tilde{f}_5$
f_5	f_5	0	$-2f_8$	0	$2f_6$	$-2f_0$	$-2f_4$	$\tilde{f}_5$	0	$2\tilde{f}_8$	0	$-2\tilde{f}_6$	$-2\tilde{f}_0$	$2\tilde{f}_4$
f_6	f_6	$-2f_7$	0	0	$-2f_5$	$2f_4$	$-2f_0$	$\tilde{f}_6$	$2\tilde{f}_7$	0	0	$2\tilde{f}_5$	$-2\tilde{f}_4$	$-2\tilde{f}_0$
$\tilde{f}_0$	$\tilde{f}_0$	$-\tilde{f}_1$	$-\tilde{f}_2$	$-\tilde{f}_3$	$-\tilde{f}_4$	$-\tilde{f}_5$	$-\tilde{f}_6$	$-f_0$	f_1	f_2	f_3	f_4	f_5	f_6
$\tilde{f}_1$	$\tilde{f}_1$	$2\tilde{f}_0$	$-2\tilde{f}_3$	$2\tilde{f}_2$	0	0	$-2\tilde{f}_7$	$-f_1$	$-2f_0$	$-2f_3$	$2f_2$	0	0	$-2f_7$
$\tilde{f}_2$	$\tilde{f}_2$	$2\tilde{f}_3$	$2\tilde{f}_0$	$-2\tilde{f}_1$	0	$-2\tilde{f}_8$	0	$-f_2$	$2f_3$	$-2f_0$	$-2f_1$	0	$-2f_8$	0
$\tilde{f}_3$	$\tilde{f}_3$	$-2\tilde{f}_2$	$2\tilde{f}_1$	$2\tilde{f}_0$	$-2\tilde{f}_9$	0	0	$-f_3$	$-2f_2$	$2f_1$	$-2f_0$	$-2f_9$	0	0
$\tilde{f}_4$	$\tilde{f}_4$	0	0	$2\tilde{f}_9$	$2\tilde{f}_0$	$2\tilde{f}_6$	$-2\tilde{f}_5$	$-f_4$	0	0	$2f_9$	$-2f_0$	$2f_6$	$-2f_5$
$\tilde{f}_5$	$\tilde{f}_5$	0	$2\tilde{f}_8$	0	$-2\tilde{f}_6$	$2\tilde{f}_0$	$2\tilde{f}_4$	$-f_5$	0	$2f_8$	0	$-2f_6$	$-2f_0$	$2f_4$
$\tilde{f}_6$	$\tilde{f}_6$	$2\tilde{f}_7$	0	0	$2\tilde{f}_5$	$-2\tilde{f}_4$	$2\tilde{f}_0$	$-f_6$	$2f_7$	0	0	$2f_5$	$-2f_4$	$-2f_0$

Table 1: Multiplication table of the vertices of a double hexagon.

Open questions

- What happens if an element is doubly alternative (its both components are alternative in the previous algebra) but does not satisfy condition $(*)$?
- What other important properties do the doubly alternative elements which satisfy condition $(*)$ possess? Presently, we have explicit formulae for their annihilators, centralizers, and orthogonalizers.
- How are orthogonality and commutativity graphs of an arbitrary real Cayley-Dickson algebra related?
- Under which conditions are two elements of a real Cayley-Dickson algebras O-equivalent (or C-equivalent), that is, have the same orthogonalizer (or centralizer)?

References

[1] D. K. Biss, D. Dugger, D. C. Isaksen, *Large Annihilators in Cayley–Dickson Algebras.* — Comm. Algebra, **36**(2) (2008), 632–664.

[2] A. E. Guterman, S. A. Zhilina, *Cayley-Dickson split-algebras: doubly alternative zero divisors and relation graphs.* — Fundam. Prikl. Mat., Accepted (2021).

[3] G. Moreno, *The zero divisors of the Cayley–Dickson algebras over the real numbers.* — Bol. Soc. Mat. Mex. (tercera serie), **4**(1) (1998), 13–28.

[4] G. Moreno, *Alternative elements in the Cayley–Dickson algebras.* — Topics in Mathematical Physics, General Relativity and Cosmology in Honor of Jerzy Plebañski, World Sci. Publ., Hackensack, NJ (2006), 333–346.

[5] S. A. Zhilina, *Orthogonality graphs of real Cayley-Dickson algebras. Part I: Doubly alternative zero divisors and their hexagons.* — Int. J. Algebra Comput., **31**(4) (2021), 663–689.

Thank you for your attention!