Computation of Certain Hurwitz Numbers

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8ECM – Applied Combinatorial and Geometric Topology

$\tilde{\Sigma}, \Sigma$ closed orientable surfaces

$f : \tilde{\Sigma} \to \Sigma$ branched cover if locally $\mathbb{C} \ni z \mapsto z^k \in \mathbb{C}$

$k > 1$: 0 is branching point

$X = \{x_1, \ldots, x_m\}$ branching points $\subset \Sigma$

$f|_{\ldots} : \tilde{\Sigma} \setminus f^{-1}(X) \longrightarrow \Sigma \setminus X$ genuine cover degree d

$f^{-1}(x_i) = \{y_{ij}\}$ $\quad \pi_i = [k_{ij}]$ partition of d

$$\boxed{\text{R-H} : \quad \chi(\tilde{\Sigma}) - (\#\pi_1 + \cdots + \#\pi_m) = d \cdot (\chi(\Sigma) - m)}$$

Branch datum : $(\tilde{\Sigma}, \Sigma, d; \pi_1, \ldots, \pi_m)$ with R-H

Q1: which branch data are realized by $f : \tilde{\Sigma} \to \Sigma$?

(not all)

Q2: "how many" f's realize a branch datum?

$f \sim g$ if $\exists \ldots$

$$\begin{tikzcd} \tilde{\Sigma} \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \tilde{\Sigma} \arrow[dl, "g"] \\ & \Sigma & \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Sigma} \arrow[r, "\phi"] \arrow[d, "f"'] & \tilde{\Sigma} \arrow[d, "g"] \\ \Sigma \arrow[r, "\psi"] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Sigma} \arrow[r, "\phi"] \arrow[d, "f"'] & \tilde{\Sigma} \arrow[d, "g"] \\ \Sigma \arrow[r, "\psi"] & \Sigma \end{tikzcd}$$

($x_1, \ldots, x_n$ fixed) $\qquad$ ϕ, ψ oriented $\qquad$ any ϕ, ψ

Fact : all different.

Today : $\gamma = \#$ for last version of $\sim$

Vast literature for Q1 (& some for Q2)

Theorem : $(\tilde{\Sigma}, \Sigma, d; \pi_1, \ldots, \pi_n)$ realizable if $\Sigma \neq S$. $\quad (\Sigma = T, 2T, \ldots)$

Conjecture : $(\tilde{\Sigma}, S, d; \pi_1, \ldots, \pi_n)$ realizable if d is a prime

Fact : enough to show for $n = 3$.

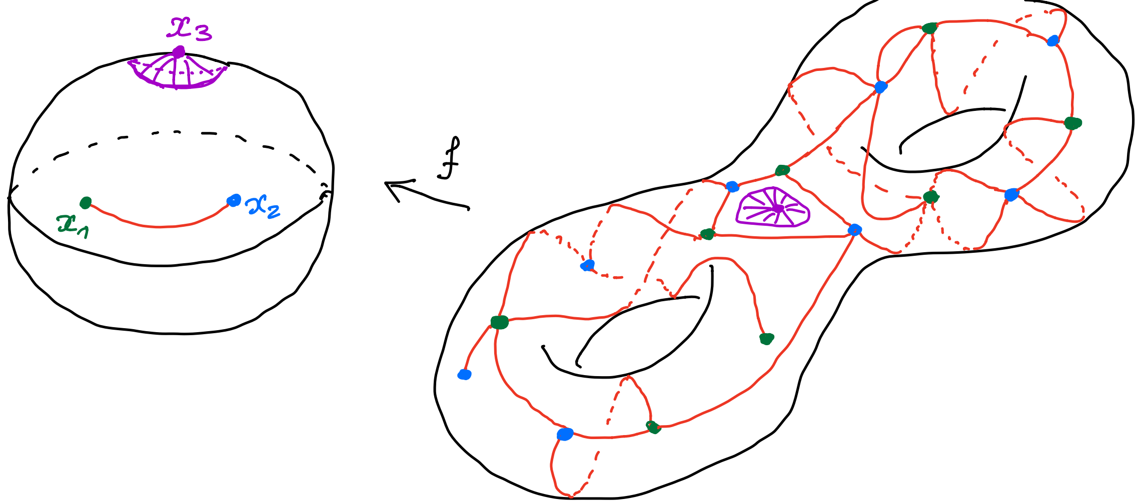
Today : $n = 3$

Computation of some

$$\nu(g, T, S, d;\ \pi_1, \pi_2, \pi_3)$$

for small g and easy π_1, π_2.

Method: dessins d'enfant



+ combinatorics, arithmetic

$$\boxed{\text{I.}\ \ d = 2k,\ n = 3,\ \pi_1 = [2, \dots, 2],\ \pi_2 = [2h+1, 1, 2, \dots, 2]}$$

$$\nu(S, S, 2k;\ [2, \dots, 2],\ [1, 1, 2, \dots, 2],\ [2k]) = 1$$

$$\nu(S, S, 2k;\ [2, \dots, 2],\ [3, 1, 2, \dots, 2],\ [p, 2k-p]) = \begin{cases} 1 & p \neq k \\ 0 & p = k \end{cases}$$

$$\nu(S, S, 2k; [2,\dots,2], [5,1,2,\dots,2], [p,q,r]) = \begin{cases} 0 & [p,q,r]=[m,m,m] \text{ or } [2m,m,m] \\ 1 & [p,q,r]=[2t,k-t,k-t] \quad \text{not as above} \\ 2 & r>k/2, \quad k-r>q>r \\ 3 & \frac{2}{3}k > r \geq \frac{k}{2}, \quad k-\frac{r}{2}>q>r \end{cases}$$

$$\nu(T, S, 2k; [2,\dots,2], [5,1,2,\dots,2], [2k]) = \left\lfloor \frac{1}{4}(k-1)^2 \right\rfloor$$

$$\nu(T, S, 2k; [2,\dots,2], [7,1,2,\dots,2], [p,2k-p])$$

$$= \begin{cases} \frac{1}{2} \cdot \left\lfloor \frac{k}{2} \right\rfloor \cdot \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) + \left\lfloor \frac{1}{4} \left\lfloor \frac{k-1}{2} \right\rfloor^2 \right\rfloor & p=k \\ \left\lfloor \frac{1}{4}(p-1)^2 \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \cdot \left(\left\lfloor \frac{p}{2} \right\rfloor - 1 \right) + (k-3)(k-p-1) & \\ \quad + \left\lfloor \frac{1}{4}\left(k - \left\lfloor \frac{p}{2} \right\rfloor - 1\right)^2 \right\rfloor - \left\lfloor \frac{k-p}{2} \right\rfloor + \left\lfloor \frac{1}{4} \left\lfloor \frac{p-1}{2} \right\rfloor^2 \right\rfloor & p \neq k \end{cases}$$

$$\nu\left(2T, S, 2k, [2,\dots,2], [9,1,2,\dots,2], [2k]\right)$$

$$= \frac{k-1}{16}\left(7k^3 - 63k^2 + 197k - 208\right) + \frac{5}{8}(5-2k)\left\lfloor \frac{k}{2} \right\rfloor$$

$$\boxed{\text{II.}\quad d = 2k,\ m = 3,\ \pi_1 = [2,\dots,2],\ \pi_2 = [2h+1, 3, 2, \dots, 2]}$$

$$\nu\left(S, S, 2k;\ [2,\dots,2], [3,3,2,\dots,2], \pi = [*,*,*]\right) = \begin{cases} 0 & k \in \pi \\ 1 & k \notin \pi \end{cases}$$

$$\nu\left(S, S, 2k;\ [2,\dots,2], [5,3,2,\dots,2], \pi = [*,*,*,*]\right) = \dots$$

$$\pi = [p,p,p,p] \longrightarrow \nu = 0$$

$$\pi = [p,p,q,q] \longrightarrow \nu = 0$$

$$\pi = [p,p,p,q] \longrightarrow \nu = \begin{cases} 0 & q = k \\ 1 & q \neq k \end{cases}$$

$$\pi = [p,p,q,r] \longrightarrow \nu = \begin{cases} 1 & r = k \text{ or } q + r = k \\ 3 & \text{otherwise} \end{cases}$$

$$\pi = [p,q,r,s] \longrightarrow \nu = \begin{cases} 2 & r + s = k \\ 3 & r = k \\ 6 & \text{otherwise} \end{cases}$$

$$\nu\left(T, S, 2k; [2,\dots,2], [3,3,2,\dots,2], [2k]\right) = \frac{1}{2}k(k-1)$$

$$\nu\left(T, S, 2k; [2,\dots,2], [5,3,2,\dots,2], [2k-p, p]\right)$$

$$= \begin{cases} 0 & p = k \\ 2\left\lfloor \frac{1}{4}(p-k-1)^2 \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \cdot (k-p-1) + \left\lfloor \frac{1}{4}(p-1)^2 \right\rfloor & \text{otherwise} \end{cases}$$

$$\nu\left(2T, S, 2k; [2,\dots,2], [7,3,2,\dots,2], [2k]\right)$$

$$= \frac{1}{48}\left(7k^4 - 70k^3 + 230k^2 - 515k + 288\right) - \frac{5}{8}(2k-5) \cdot \left\lfloor \frac{k}{2} \right\rfloor$$

$\boxed{\text{III. } d = 2k+1, \ n = 3, \ \pi_1 = [1,2,\dots,2], \ \pi_2 = [2k+1, 2, \dots, 2]}$

$$\nu\left(S, S, 2k+1; [1,2,\dots,2], [1,2,\dots,2], [2k+1]\right) = 1$$

$$\nu\left(S, S, 2k+1; [1,2,\dots,2], [3,2,\dots,2], [2k+1-p, p]\right) = 1$$

$$\nu\left(S, S, 2k+1; [1,2,\dots,2], [5,2,\dots,2], \pi = [*,*,*]\right) = \cdots$$

$$\pi = [p,p,p] \longrightarrow \nu = 0$$

$$\pi = [p,p,q] \longrightarrow \nu = 1$$

$$\pi = [p,q,r] \longrightarrow \nu = \begin{cases} 2 & p > k \\ 3 & \text{otherwise} \end{cases}$$

$$\nu\left(T, S, 2k+1; [1,2,\dots,2], [5,2,\dots,2], [2k+1]\right) = \left\lfloor \left(\frac{k}{2}\right)^2 \right\rfloor$$

$$\nu\left(T, S, 2k+1; [1,2,\dots,2], [7,2,\dots,2], [2k+1-p, p]\right) \qquad (p > k)$$

$$= \left\lfloor \left(\frac{1}{2}\left(k - \left\lfloor \frac{p-1}{2} \right\rfloor\right)\right)^2 \right\rfloor + \left\lfloor \left(\frac{1}{2}\left\lfloor \frac{p-1}{2} \right\rfloor\right)^2 \right\rfloor$$

$$+ \left\lfloor \frac{p}{2} \right\rfloor^2 - (p-1) \cdot \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \left(\frac{p}{2}\right)^2 \right\rfloor$$

$$+ k^2 - k(p-1) + \frac{1}{2}(p-1)(p-4)$$

except for $k=4, p=7$ where $\nu = 5$ (not 6)

$$\gamma\left(2T, S, 2k+1; [1,2,\dots,2], [9,2,\dots,2], [2k+1]\right)$$

$$= \frac{k}{16}\left(7k^3-42k^2+72k-37\right) + \frac{5}{8}(2k-3)\left\lfloor \frac{k}{2} \right\rfloor$$